J. K. Aggarwal (S'62–M'65–SM'74–F'76) was born in Amritsar, India. He received the B.S. degree in mathematics and physics from the University of Bombay, Bombay, India, in 1956, the B. Eng. degree from the University of Liverpool, Liverpool, England, in 1960, and the M.S. and Ph.D. degrees from the University of Illinois, Urbana, IL, in the years 1961 and 1964, respectively.

He joined the University of Texas in 1964 as an Assistant Professor and has since held positions as Associate Professor in 1968 and Professor in 1972. Currently, he is a Professor of Electrical Engineering and of Computer Science at The University of Texas at Austin, Austin. Further, he was a Visiting Assistant Professor to Brown University, Providence, RI, in 1968 and a Visiting Associate Professor at the University of California, Berkeley, during 1969–1970. He has published numerous technical papers and a text book, *Notes on Nonlinear Systems*, Published by Van Nostrand Reinhold, in 1972, in the series Notes on System Sciences. He was Associate Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS and Coeditor of the Special Issue on Digital Filtering and Image Processing, March 1975. He was also Editor of the Circuits and Systems Society Newsletter. His current research interests are digital filters, computational methods, and image processing.

Dr. Aggarwal is an ADCOM member of the IEEE Circuits and Systems Society and Chairman of the Technical Committee on Signal Processing. He is also a member of Sigma Xi, Eta Kappa Nu, and Rotary International.

Variable Cutoff Linear Phase Digital Filters

ALAN V. OPPENHEIM, SENIOR MEMBER, IEEE, WOLFGANG F. G. MECKLENBRÄUKER, AND RUSSELL M. MERSEREAU, MEMBER, IEEE

***Abstract*—This paper discusses an approach to the implementation of a linear phase finite impulse response filter for which the cutoff frequency is controlled through a small number of parameters. The approach is based on a transformation implemented by replacing a subnetwork in a prototype network.**

I. Introduction

IT IS OFTEN OF INTEREST to implement in hardware a digital filter for which the cutoff frequency is variable. One possible approach is to vary all of the filter coefficients in such a way that the cutoff frequency varies in the desired manner. This, of course, requires the ability to vary a number of parameters. Furthermore, the filter coefficients are generally a complicated function of the filter cutoff frequency. This procedure may perhaps be practical when we wish to vary the filter cutoff frequency only occasionally. It would generally be more desirable, however, to construct the filter in such a way as to permit the cutoff frequency to be controlled by only a single parameter.

One approach to implementation of a variable cutoff digital filter in which the cutoff frequency could be controlled through a single parameter was suggested by Schuessler and Winkelnkemper [1]. In their approach each of the delay elements in the structure for a prototype filter is replaced by a first-order all pass network. This has the effect of replacing the delay operator by an all pass transformation in the filter transfer function. The frequency response of the transformed filter is then identical to the frequency response of the prototype filter on a distorted frequency scale. As the parameter of the all pass network is varied, the distortion of the frequency axis is varied, and thus so is the filter cutoff frequency. The use of this procedure is restricted to a finite impulse response (FIR) prototype filter since for an IIR prototype filter a structure with delay-free loops results. When the all pass transformation is applied to an FIR prototype filter the resulting filter has an impulse response of infinite length due to the fact that the all pass network is recursive. Furthermore, even if the prototype filter has linear phase, the phase of the transformed filter will be nonlinear.

In some applications it may be desirable and important to implement a variable cutoff filter for which the impulse response is of finite length and the phase is linear if a linear phase FIR prototype filter is used. In this paper we discuss a class of transformations for which these proper-

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A. V. Oppenheim and R. M. Mersereau are with the Department of Electrical Engineering and Computer Science and the Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge.

W. F. G. Mecklenbräuker was a Visiting Scientist in the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge. He is now with the Philips Research Laboratories, Eindhoven, The Netherlands.

ties of the prototype filter are preserved. In the next section we consider the class of transformations and in Section III we discuss the resulting network structures for implementation of the variable cutoff filters.

II. Frequency Transformations for Linear Phase Variable Filters

Consider a causal linear phase FIR filter with an impulse response $h(n)$ of length $2N+1$. Any linear phase filter of this type can be expressed in the form

$$h(n)=h_0(n-N) \tag{1}$$

where $h_0(n)$ is the impulse response of a zero phase FIR filter which is symmetric, i.e.,

$$h_0(n)=h_0(-n). \tag{2}$$

From (1) and (2), it follows that $H(z)$, the transfer function of the linear phase filter can be expressed as [2]

$$H(z)=z^{-N}H_0(z) \tag{3a}$$

where

$$H_0(z)=h_0(0)+\sum_{n=1}^{N}h_0(n)\left[z^n+z^{-n}\right]. \tag{3b}$$

Since terms of the form (z^n+z^{-n}) can be expressed in the form

$$z^n+z^{-n}=2T_n\left[\frac{z+z^{-1}}{2}\right]$$

where $T_n(x)$ is a Chebyschev polynomial of nth order, $H_0(z)$ can be rewritten as

$$H_0(z)=\sum_{n=0}^{N}a(n)\left[\frac{z+z^{-1}}{2}\right]^n \tag{4}$$

where the coefficients $a(n)$ are related to $h_0(n)$ through the coefficients of the Chebyshev polynomials. The frequency response of the linear phase filter is thus

$$H(e^{j\omega})=e^{-j\omega N}H_0(e^{j\omega}) \tag{5a}$$

where

$$H_0(e^{j\omega})=\sum_{n=0}^{N}a(n)(\cos\omega)^n. \tag{5b}$$

The basic approach to obtaining a variable cutoff linear phase filter is to apply a transformation to $H_0(z)$ which preserves the frequency characteristics but distorts the frequency axis. Specifically, let us consider a frequency response $\hat{H}_0(e^{j\Omega})$ obtained from $H_0(e^{j\omega})$ by the substitution of variables

$$\cos\omega=\sum_{k=0}^{P}A_k(\cos\Omega)^k. \tag{6}$$

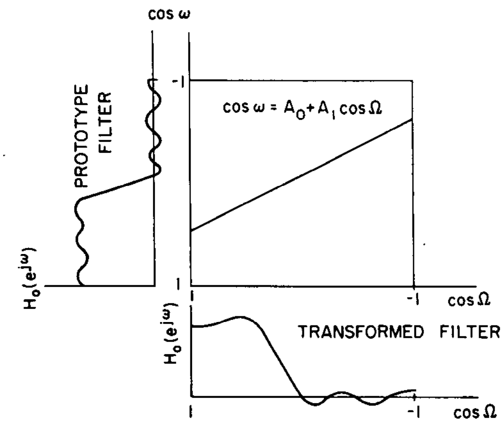

Fig. 1. First-order frequency transformation.

Since $\hat{H}_0(e^{j\Omega})$ is still expressible as a cosine polynomial, the corresponding unit sample response is still symmetrical. However, it is now of length $2NP+1$. The transfer function $\hat{H}(z)$ corresponding to the causal linear phase filter is then

$$\hat{H}(z)=z^{-NP}\hat{H}_0(z). \tag{7}$$

As the coefficients A_k in the transformation of (6) are varied the relationship between the prototype frequency response $H_0(e^{j\omega})$ and the transformed frequency response $\hat{H}_0(e^{j\Omega})$ varies. By appropriately constraining the coefficients A_k the cutoff frequency, transition width, etc., can be varied. In order to guarantee that the transformation of (6) represents a mapping of $H_0(z)$ for z on the unit circle, the coefficients in (6) must be constrained such that for $-\pi<\Omega<\pi, |\cos\omega|\leqslant 1$.

A particularly useful case of the transformation (6) appears to be the first-order transformation of the form

$$\cos\omega=A_0+A_1\cos\Omega. \tag{8}$$

The resulting mapping is illustrated in Fig. 1. This transformation is similar to that used by Siegel [3] and by Rabiner and Herrmann [4] to analytically relate optimum FIR filter designs. For the case of a variable low pass filter we may wish to constrain the transformation so that

$$\hat{H}_0(e^{j\Omega})\big|_{\Omega=0}=H(e^{j\omega})\big|_{\omega=0} \tag{9}$$

in which case we require that $A_0+A_1=1$. The substitution thus takes the form

$$\cos\omega=A_0+(1-A_0)\cos\Omega \tag{10}$$

with the constraint that

$$0\leqslant A_0<1 \tag{11}$$

in order to ensure that $|\cos\omega|\leqslant 1$. If the prototype filter is low pass with a cutoff frequency of ω_c then the transformed filter will have a cutoff frequency Ω_c where

$$\Omega_c=\cos^{-1}\left[\frac{\cos\omega_c-A_0}{1-A_0}\right]. \tag{12}$$

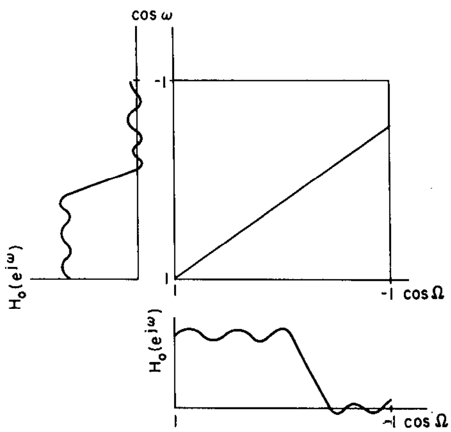

Fig. 2. First-order frequency transformation with $A_0+A_1=1$.

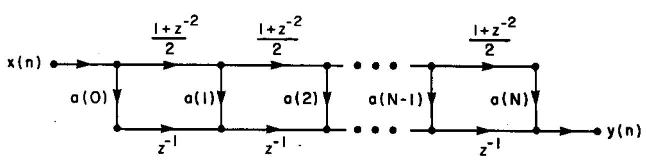

Fig. 3. Implementation of linear phase FIR filter.

The relationship between the prototype filter and the transformed filter with the constraint of (9) is illustrated in Fig. 2. We observe that with a low pass prototype filter as A_0 varies between zero and unity the cutoff frequency of the transformed filter is always greater than or equal to the cutoff frequency of the prototype filter. The reverse situation can be obtained by choosing $A_1=(1+A_0)$ so that

$$\hat{H}_0(e^{j\Omega})\big|_{\Omega=\pi}=H(e^{j\omega})\big|_{\omega=\pi}. \tag{13}$$

In this case the transformation is given by

$$\cos\omega=A_0+(1+A_0)\cos\Omega \tag{14}$$

with the constraint that

$$-1<A_0\leqslant 0.$$

For a low pass prototype filter, as A_0 varies, the cutoff frequency of the transformed filter will now be less than or equal to the cutoff frequency of the prototype filter. In more general situations we may wish to specify a different relationship between the parameters A_0 and A_1. For example, for a bandpass prototype filter we may want to constrain A_0 and A_1 such that the center frequency of the filter remains fixed and only the bandwidth is varied.

It should be stressed that the use of the transformation of (6) or (8) is directed at the *implementation* of a digital filter for which the cutoff frequency is easily varied through a small number of parameters. In general, even when the prototype filter is an optimal filter [2], the filter resulting after the transformation of (6) or (8) will not be optimal. However, a variable cutoff filter which is always optimal and for which the cutoff frequency varies over a reasonable range can, in general, only be implemented by changing all of the filter coefficients for each setting of the filter cutoff frequency.

III. Network Structures for Implementation of Linear Phase Variable Cutoff Filters

Taylor Structure

From (3a) and (4), the transfer function of the causal linear phase prototype filter can be expressed in the form

$$H(z)=z^{-N}\sum_{n=0}^{N}a(n)\left[\frac{z+z^{-1}}{2}\right]^n \tag{15a}$$

$$=\sum_{n=0}^{N}a(n)z^{-N+n}\left[\frac{1+z^{-2}}{2}\right]^n. \tag{15b}$$

A direct implementation of $H(z)$, as expressed in (15b), is shown in Fig. 3 in signal flow graph notation, where each of the branches with transmittance $(1+z^{-2})/2$ would be implemented by a subnetwork. The resulting structure corresponds to a special case of the Taylor structure [5] for linear phase FIR filters. We note that each of the subnetworks has a maximum gain of unity, which is generally convenient for scaling purposes. Now, let $\hat{H}_0(Z)$ denote the transfer function of the zero phase transformed filter and $H_0(z)$ the transfer function of the zero phase prototype filter. From (6), $\hat{H}_0(Z)$ and $H_0(z)$ are related through the substitution

$$\frac{z+z^{-1}}{2}=\sum_{k=0}^{P}A_k\left(\frac{Z+Z^{-1}}{2}\right)^k \tag{16}$$

and consequently, combining (4), (7), and (16),

$$\hat{H}(Z)=Z^{-NP}\sum_{n=0}^{N}a(n)\left[\sum_{k=0}^{P}A_k\left(\frac{Z+Z^{-1}}{2}\right)^k\right]^n \tag{17}$$

which can be rewritten as

$$\hat{H}(Z)=\sum_{n=0}^{N}a(n)Z^{-P(N-n)}\left[\sum_{k=0}^{P}A_kZ^{k-P}\left(\frac{1+Z^{-2}}{2}\right)^k\right]^n \tag{18}$$

or

$$\hat{H}(Z)=\sum_{n=0}^{N}a(n)Z^{-P(N-n)}\left[\hat{H}_P(Z)\right]^n \tag{19a}$$

where

$$\hat{H}_P(Z)=\sum_{k=0}^{P}A_KZ^{k-P}\left(\frac{1+Z^{-2}}{2}\right)^k. \tag{19b}$$

An implementation of the transfer function (19) is depicted in Fig. 4. We observe that it is the same general configuration as in Fig. 3 with the sections with transfer function $(1+z^{-2})/2$ replaced by $\hat{H}_P(Z)$ and the first-order delay branches replaced by Pth-order delays. The

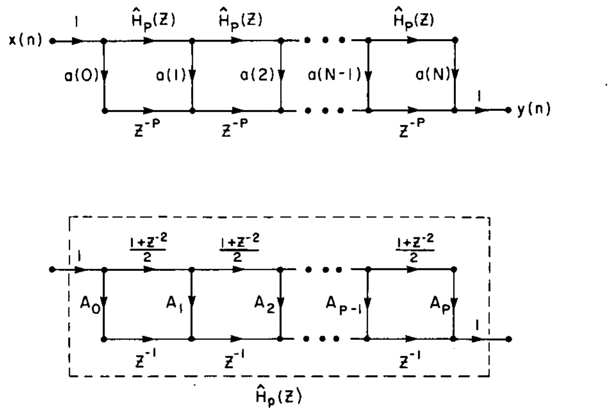

Fig. 4. Implementation of linear phase variable cutoff filter with Pth-order transformation.

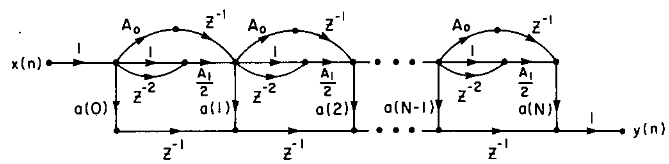

Fig. 5. Implementation of linear phase variable cutoff filter with first-order transformation.

characteristics of the transformed filter are then varied by changing the coefficients $A_0, A_1, \cdots, A_P$. We note that since the impulse response of the transformed filter is of length $2NP+1$ the minimum number of delays required is $2NP$. Consequently, the structure of Fig. 4 is not canonic in the number of delays required.[1]

For the first-order transformation, i.e., $P=1$, (19) becomes

$$\hat{H}(Z)=\sum_{n=0}^{N} a(n)Z^{-(N-n)}\left[A_0Z^{-1}+A_1\left(\frac{1+Z^{-2}}{2}\right)\right]^n \tag{20}$$

and the network of Fig. 4 reduces to that shown in Fig. 5. Each of the networks presented in Figs. 3 through 5 can be rearranged slightly with resulting tradeoffs in delay registers, modularity, and scaling requirements. The ones presented are intended to be generally indicative of the way in which transformations of the form of (6) can be incorporated into the filter structure.

Cascade Form Structure

As an alternative to the Taylor structure discussed above the linear phase prototype filter can be implemented as a cascade of the fourth-order Taylor sections

[1]In the networks of Figs. 3 and 4 the number of delays required can be reduced slightly. In Fig. 3, for example, the first two delays fed by the output of the coefficient branches $a(0)$ and $a(1)$ can be shared with the delays in the first two networks with transfer function $(1+z^{-2})/2$. This results in only a modest reduction in delay elements at the expense of modularity in the overall structure.

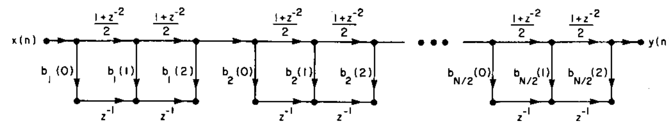

Fig. 6. Cascade form implementation of linear phase FIR filter using fourth-order Taylor sections.

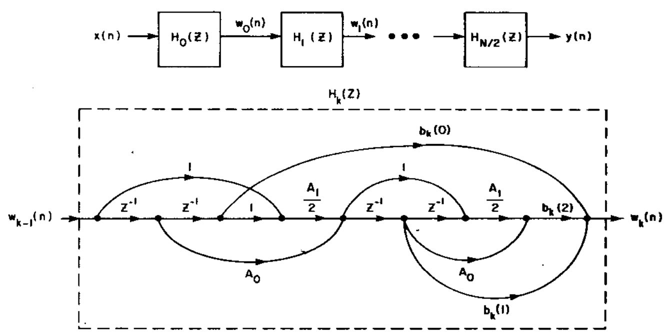

Fig. 7. Canonic implementation of linear phase variable cutoff filter with first-order transformation.

and the transformation of (16) applied. Specifically, $H(z)$ as given by (15a) can be factored into the form

$$H(z)=\prod_{k=1}^{N/2}\left[z^{-2}\sum_{n=0}^{2} b_k(n)\left[\frac{z+z^{-1}}{2}\right]^n\right] \tag{21a}$$

$$=\prod_{k=1}^{N/2}\left[\sum_{n=0}^{2} b_k(n)z^{-2+n}\left[\frac{1+z^{-2}}{2}\right]^n\right]. \tag{21b}$$

The resulting filter structure corresponds to a cascade of fourth-order Taylor sections of the form of Fig. 3 as depicted in Fig. 6. To obtain the variable cutoff linear phase filter, each fourth-order section is transformed in the same manner in which the Taylor network of Fig. 3 was transformed to obtain the network of Fig. 4. For the general transformation of (16) the resulting variable cutoff filter is not canonic in the number of delays. For the first-order transformation, however, with $P=1$ the transformed network can be rearranged to be canonic in the number of delays. The resulting structure is depicted in Fig. 7.

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Alan V. Oppenheim (S'57–M'65–SM'71) received the S.B. and S.M. degrees in 1961 and the Sc.D. degree in 1964, all in electrical engineering, from the Massachusetts Institute of Technology, Cambridge.

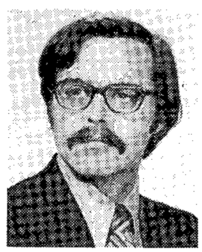
From 1961 to 1964 he was a member of the M.I.T. Research Laboratory of Electronics and an Instructor in the Department of Electrical Engineering. During this period his research activities involved the application of modern algebra to the characterization of nonlinear systems and the development of homomorphic filtering as an approach to some nonlinear filtering problems. In 1964, he became an Assistant Professor in the M.I.T. Department of Electrical Engineering and a Staff Member of the Research Laboratory of Electronics. In 1969 he was appointed Associate Professor, after spending 1967 to 1969 as a Staff Member of M.I.T. Lincoln Laboratory, Lexington. During the academic year 1972–1973, he was a Guggenheim Fellow on sabbatical leave at the University of Grenoble, Grenoble, France. He is coauthor of a text on *Digital Signal Processing*. His present research activities are in the area of digital signal processing and its application to speech and image processing.

Dr. Oppenheim is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi. He is Chairman of the Digital Signal Processing Committee of the IEEE Group on Acoustics, Speech, and Signal Processing.

Wolfgang F. G. Mecklenbräuker was born in Dortmund, Germany, on June 16, 1938. He received the Dipl.-Ing. degree in electrical engineering from Aachen University, Aachen, Germany, in 1964, and the Dr.-Ing. degree in electrical engineering from Darmstadt University, Darmstadt, Germany, in 1969.

He was a Research Associate at Darmstadt University from 1964 to 1970, where he worked on network theory and system theory. In 1971 he joined the Philips Research Laboratories, Eindhoven, The Netherlands. Currently, he is on sabbatical leave at the Massachusetts Institute of Technology, Cambridge. His research interests are in network theory and digital signal processing.

Dr. Mecklenbräuker is a member of the Nachrichtentechnische Gesellschaft (NTG), Germany.



Russell M. Mersereau (S'69–M'73) was born in Cambridge, MA., on August 29, 1946. He received the S.B., S.M., and Sc.D. degrees from the Massachusetts Institute of Technology, Cambridge, in 1969, 1969, and 1973, respectively.

From 1971 to 1973, he was an Instructor in the Department of Electrical Engineering at Massachusetts Institute of Technology, and since 1973 he has been with the Research Laboratory of Electronics as a Research Associate where he is concerned with the digital signal processing of multidimensional signals.

Dr. Mersereau is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.