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Signal synthesis and reconstruction from partial Fourier-domain information

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We review and interpret previous results of Fourier synthesis of a signal from its partial Fourier-domain information. Specifically, the high intelligibilities of the phase-only, signed-magnitude-only, and one-bit–phase-only signals are shown to be closely related to one another. In addition, we review and interpret previous results and develop new results on exact reconstruction of a signal from its partial Fourier-domain information.

1. INTRODUCTION

In a variety of practical problems, only partial Fourier-domain information is available about a signal, and it is desired either to reconstruct the signal exactly or to synthesize a signal that in some sense has some intelligibility consistent with the original signal. Over the past several years a number of results related to this problem have been developed in the Digital Signal Processing Group at MIT. In this paper, we summarize these previously reported results and in Section 4 present a number of new results and interpretations related to one aspect of the problem.

In broad terms, the partial Fourier-domain information that we have assumed to be available is the Fourier-transform (FT) phase alone, the FT magnitude alone, or, as we define in more detail in Section 2, the signed FT magnitude corresponding to magnitude together with a bipolar function (± 1) representing one bit of phase information at each frequency. The attempt to reconstruct a signal exactly from FT-magnitude information is commonly referred to in the literature as the phase-retrieval problem.[1] Correspondingly, reconstruction from FT phase alone is typically referred to as the magnitude-retrieval problem. Although both of these are of potential practical importance, it is generally acknowledged that the phase-retrieval problem is more common since there are many situations in areas such as electron microscopy,[2] x-ray crystallography,[3] and optical astronomy[4] in which the magnitude or the intensity of a diffraction pattern or an interference pattern is recorded and from which it is hoped that more-complete information can be recovered. Although the companion situation, in which only the FT phase is available, is perhaps of less practical significance, there are important situations in which it arises. One is when a signal is distorted by filtering with a symmetrical, zero-phase blurring or point-spread function, as could occur in defocusing in some optical systems. In this case, the magnitude is degraded in an unknown way, but the phase is retained undegraded. Li and Kurkjian[5] have also shown how some problems in arrival-time estimation can be formulated as a magnitude-retrieval problem. Specifically, a situation encountered in applications such as seismics and ocean acoustics is that of an unknown wavelet propagating nondispersively in a reverberatory or multipath environment. Two receivers placed in such an environment will each record the arrival of this wavelet numerous times, each time with a different attenuation factor. The method proposed for estimating the arrival times and attenuations at each receiver corresponds to constructing a finite-length sequence whose phase equals that of the cross spectrum of the two received signals and consequently is known. This information, together with the finite-length constraint, is then used to recover the original sequence. An additional problem in which only phase information is available arises in the field of paleomagnetism, in which, by an examination of core samples from the earth, a history of the direction, but not of the strength, of the magnetic field over time can be obtained. It would be highly desirable to be able to estimate or recover from this phase or direction history the associated strength or magnitude of the original magnetic field.

In all the problems above, and in the discussion that follows, there are two distinct sets of issues. As we discuss in Section 2, it is often possible to synthesize a signal in a more-or-less straightforward way from partial information that captures much of the intelligibility of the signal, even if it is not an accurate reconstruction on a point-by-point basis. A separate and distinctly different question and set of issues relate to the conditions under which exact reconstruction is possible from the partial information and the algorithmic issues associated with this reconstruction. Our previous work on this is summarized in Section 3, and some new interpretations and results related to one aspect of the topic are developed in Section 4.

In developing the discussion in the remainder of the paper, we consider both one-dimensional and multidimensional signals and their FT's. Notationally, a general N-dimensional signal will be denoted by $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is the vector of independent variables, and its associated N-dimensional Fourier transform by $F(\boldsymbol{\omega})$, where $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_N)$. A one-dimensional signal and its associated FT are denoted by $f(x)$ and $F(\omega)$, respectively. When the discussion applies explicitly to discrete-time signals, the signal is denoted by $f(\mathbf{n})$.

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The FT is, of course, a complex-valued function, which we express in Cartesian form as

$$F(\omega) = F_r(\omega) + jF_i(\omega), \quad (1)$$

where $F_r(\omega)$ and $F_i(\omega)$ are the real and the imaginary parts, respectively, and in polar form as

$$F(\omega) = M_f(\omega)\exp[j\theta_f(\omega)], \quad (2)$$

where $M_f(\omega)$ and $\theta_f(\omega)$ are the magnitude and the phase, respectively, and $-\pi < \theta_f(\omega) \leq \pi$. In this representation, of course, $M_f(\omega)$ is always real and nonnegative.

We also make reference to a representation to which we refer as a signed-magnitude and phase representation, corresponding to signed magnitude and its generalization. In this representation, we express $F(\omega)$ in the form

$$F(\omega) = A_f(\omega)\exp[j\psi_f(\omega)], \quad (3)$$

where $\psi_f(\omega)$ is restricted to the interval $-(\pi/2) \leq \psi_f(\omega) < (\pi/2)$ and

$$Af(\omega) = M_f(\omega)\mathrm{sign}[F_r(\omega)]. \quad (4)$$

For example, if $F(\omega)$ is real but not necessarily positive in the magnitude and phase representation of Eq. (2), $\theta_f(\omega)$ is zero or π, depending on the sign of $F(\omega)$, whereas, in the signed-magnitude and phase representation of Eq. (3), $A_f(\omega) = F(\omega)$ and $\psi_f(\omega) = 0$.

We also find it convenient to consider a generalized form of the representation in Eq. (3). This generalized representation is

$$F(\omega) = A_f{}^{\alpha}(\omega)\exp[j\psi_f{}^{\alpha}(\omega)] \quad (5)$$

where α is a specified constant in the range

$$0 < \alpha < \pi$$

and

$$A_f{}^{\alpha}(\omega) = M_f(\omega)S_f{}^{\alpha}(\omega), \quad (6)$$

with

$$S_f{}^{\alpha}(\omega) = \mathrm{sign}\left(\mathrm{Re}\left\{\exp\left[j\left(\frac{\pi}{2} - \alpha\right)\right]F(\omega)\right\}\right). \quad (7)$$

Equivalently,

$$S_f{}^{\alpha}(\omega) = \begin{cases} 1, & \alpha - \pi < \theta_f(\omega) \leq \alpha \\ -1, & \text{otherwise} \end{cases}. \quad (8)$$

$A_f{}^{\alpha}(\omega)$ is again referred to as the signed magnitude. For $\alpha = \pi/2$, the representation of Eq. (5) reduces to the representation of Eq. (3).

The bipolar function $S_f{}^{\alpha}(\omega)$ incorporates one bit of phase information at each frequency. Thus the complex plane is divided into two regions separated by a straight line passing through the origin and at an angle α with the real axis. The one-bit phase information corresponds to knowing in which part of the complex plane $F(\omega)$ lies for each ω.

2. FOURIER SYNTHESIS FROM PARTIAL INFORMATION

A. Phase-Only and Magnitude-Only Fourier Synthesis

When either the FT phase alone or the FT magnitude alone is available, one can consider synthesizing the signal by combining the known information with an average assumption about the remainder of the transform. Combining the correct phase with either a constant or an average magnitude is referred to as phase-only Fourier synthesis, and the result is denoted as $f_p(\mathbf{x})$. Combining the correct magnitude with zero phase is referred to as magnitude-only Fourier synthesis, and the result is denoted as $f_m(\mathbf{x})$.

Apparently independently and in a number of different contexts, it has been recognized that many features of a signal are retained in a phase-only Fourier synthesis but not in a magnitude-only Fourier synthesis. This observation about phase applies to one-dimensional, two-dimensional, and three-dimensional signals. For example, both phase-only and magnitude-only acoustical and optical holograms have been studied.[6–9] For phase-only holograms (also referred to as kinoforms), only the phase of the scattered wave front is recorded and the magnitude is assumed to be constant, whereas in the magnitude-only hologram the phase is assumed to be zero and only the magnitude of the scattered wave front is recorded. In general, with reconstruction from magnitude-only holograms, the reconstructed object is not of much value in representing the original object, whereas reconstructions from phase-only holograms have many important features in common with the original objects.

Closely related to phase-only and magnitude-only holograms are phase-only and magnitude-only images. As with kinoforms, a phase-only image has a FT phase equal to that of the original image and a FT magnitude of unity or perhaps more generally representative of the spectral magnitude of images, such as the average over an ensemble of unrelated images. Many of the features of the original image are clearly identifiable in the phase-only image but not in the magnitude-only image. This is illustrated in Fig. 1,[10] in which we show an original image, the phase-only image with unity magnitude, and the phase-only image with average magnitude. The magnitude-only image (not shown) has most of its energy concentrated at the origin because the phase is zero and has no easily recognizable features in common with the original.

Similar observations have also been made in the context of speech signals. Specifically, we previously demonstrated that the intelligibility of speech is retained if the phase of the FT of a long segment of speech is combined with unity magnitude, whereas the intelligibility is not retained if the magnitude of the FT is combined with zero phase.[11,12]

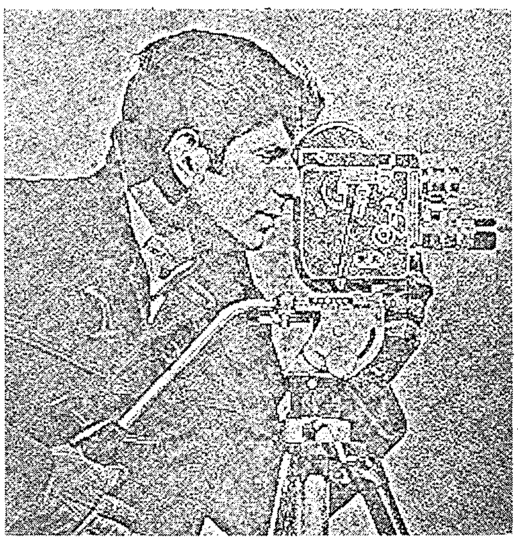
B. Fourier Synthesis from $A_f{}^{\alpha}(\omega)$ or $S_f{}^{\alpha}(\omega)$

Closely related to phase-only and magnitude-only synthesis is the Fourier synthesis from signed magnitude alone. In this case the synthesized signal, denoted by $f_a{}^{\alpha}(\mathbf{x})$, is obtained as the inverse FT of $A_f{}^{\alpha}(\omega)$, corresponding to $\psi_f{}^{\alpha} = 0$ in Eq. (3). This then corresponds to a synthesis with the correct Fourier magnitude and one bit of phase information. In Fig. 2 is shown[13] the signed-magnitude-only image of the original image of Fig. 1(a), with $\alpha = \pi/2$. Clearly, much of the original intelligibility is preserved in the synthesis from signed magnitude alone.

As was suggested to us by Gassman,[14] intelligibility of $f_a{}^{\pi/2}(\mathbf{x})$ is closely related to the intelligibility of $f_p(\mathbf{x})$. In particular, let us restrict $f(\mathbf{x})$ to have a one-sided region of support, i.e.,

$$f(\mathbf{x}) = 0 \qquad \text{outside } \mathbf{x} \geq 0 \quad (9)$$

(a)



(b)

(c)

Fig. 1. (a) Original image of 256 × 256 pixels with 8 bits/pixel. (b) Phase-only image with unity magnitude. (c) Phase-only image with average magnitude.

Fig. 2. Signed-magnitude-only image with average magnitude.

and define $g(\mathbf{x})$ as

$$g(\mathbf{x}) = f(\mathbf{x}) + f(-\mathbf{x}). \tag{10}$$

Then $G(\boldsymbol{\omega}) = 2\,\mathrm{Re}[F(\boldsymbol{\omega})]$, and the phase of $G(\boldsymbol{\omega})$ represents $\mathrm{sign}\{\mathrm{Re}[F(\boldsymbol{\omega})]\}$. Consequently, $g_p(\mathbf{x})$, the phase-only synthesis of $g(\mathbf{x})$, and $f_a{}^{\pi/2}(\mathbf{x})$, the signed-magnitude-only synthesis of $f(\mathbf{x})$, have identical phase and differ only in FT magnitude. Based on intelligibility of phase-only synthesis, $g_p(\mathbf{x})$ can be expected to retain much of the intelligibility of $g(\mathbf{x})$, and this in fact has been verified by a number of experiments. For example, with $f(\mathbf{x})$ corresponding to an image, $g(\mathbf{x})$ is the concatenation of the image with itself reflected with respect to the origin. Experimentally, $g_p(\mathbf{x})$ has the same symmetry, retaining much of the intelligibility of $g(\mathbf{x})$. Figure 3 shows the first quadrant of $g_p(\mathbf{x})$ when $f(\mathbf{x})$ is the original image of Fig. 1(a). A similar effect has also been observed for speech signals.

Since the phase of $G(\boldsymbol{\omega})$ represents $\mathrm{sign}\{\mathrm{Re}[F(\boldsymbol{\omega})]\}$, which is one bit of phase information of $F(\boldsymbol{\omega})$, $g_p(\mathbf{x})$ represents the signal synthesized by combining one bit of phase of $F(\boldsymbol{\omega})$ with a constant magnitude and is therefore referred to as a one-

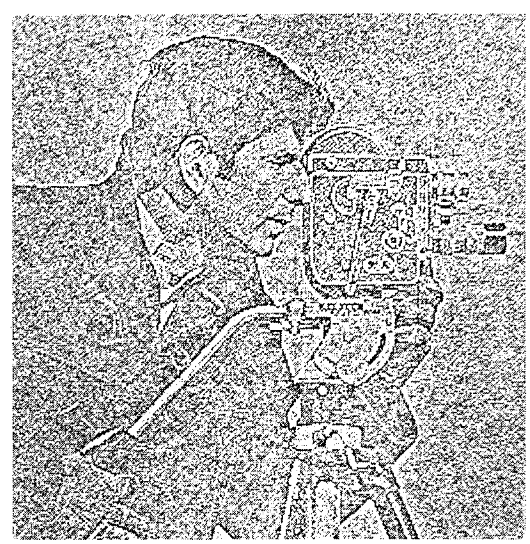


Fig. 3. One-bit phase-only image with constant magnitude.

Fig. 4. One-bit phase-only image with average magnitude.

bit–phase-only signal. The image shown in Fig. 3, then, is the one-bit–phase-only synthesis of the original image of Fig. 1(a). In the phase-only synthesis, the correct FT phase has been combined either with a constant magnitude or with a representative average magnitude. Similarly, in the one-bit–phase-only synthesis, the one bit of phase information can be combined with an average magnitude. An example of this is shown in Fig. 4, in which the one-bit phase information obtained from the original image of Fig. 1(a) was combined with an average magnitude.

A common piece of information used in both the phase-only and the signed-magnitude-only synthesis is the one bit of phase information. This suggests that the high intelligibility of phase-only and signed-magnitude-only synthesis is perhaps due primarily to the high intelligibility of the one-bit–phase-only synthesis. This speculation is supported in part by the low degree of intelligibility preserved in the magnitude-only signal and in part by the low degree of intelligibility preserved when one bit of phase information is removed from the phase-only signal. Specifically, when the one bit of phase information is removed from the correct phase by retaining the correct phase for half of the phase and by adding π to half of the remaining phase, and then when the signal is reconstructed by combining the modified phase with a constant or average magnitude, the resulting signal has been observed to retain little intelligibility.

3. EXACT RECONSTRUCTION FROM PARTIAL INFORMATION

A. Exact Reconstruction from Phase

The reasonably high intelligibility of phase-only signals demonstrates the fact that much of the important information resides in the phase. Although, in general, a signal is not uniquely defined by its FT phase, it may be under certain conditions or constraints. One well-known set of conditions under which a signal may be uniquely recovered to within a scale factor from its phase is the minimum-phase or maximum-phase condition. There are also other, different sets of conditions unrelated to the minimum-phase or maximum-phase condition under which signal recovery to within a scale factor is possible from the phase.

In the context of our research program, it has been shown that, if a one-dimensional discrete-time signal is of finite length and has a z transform with no zeros on the unit circle or in conjugate reciprocal pairs, then phase information alone is sufficient for signal reconstruction.[15,16] This result has also been extended in a number of ways to multidimensional signals[15–17] and to reconstruction when the phase is known only at a set of sample frequencies.

A variety of algorithms have also been developed for implementing exact signal reconstruction from phase. One algorithm consists of solving a set of simultaneous linear equations. A second, more robust algorithm is an iterative procedure that imposes alternately the finite-length constraint in the time domain and the known phase information in the frequency domain.[15,18]

B. Reconstruction from Fourier-Transform Magnitude

It is well known that the above conditions for signal reconstruction from FT phase do not also apply to reconstruction from FT magnitude. Theoretically, a two- or higher-dimensional finite-length signal can be recovered to within a translation, reflection with respect to the origin, and a sign, from samples of FT magnitude,[16] when its z transform is not factorable, which is satisfied in most cases of practical interest. The procedure to recover a signal from its FT magnitude, however, does not appear to be robust, and practical algorithms have not been developed.

Recently, we showed that one context in which FT magnitude is sufficient is when it is the magnitude of the *short-time* FT that is available. In many application areas, signal processing is carried out on the basis of a short-time Fourier analysis. In speech processing, in particular, the short-time FT is used in a wide variety of applications as the basis for both speech analysis and speech synthesis. Often it is the FT magnitude that is recorded or processed under the assumption that the loss of information associated with discarding the phase is acceptable. The theory that has been developed demonstrates that, under mild conditions, the short-time FT magnitude is sufficient for exact representation of the signal.[19,20] In essence, the requirement is that the analysis window be known and that the short-time FT magnitude be available at time increments that are less than one half of the length of the analysis window. Based on these conditions, the original signal can be exactly recovered to within a multiplication by plus or minus unity. Furthermore, a variety of algorithms implementing this reconstruction have been developed and implemented.

The importance of this theory relates not only to reconstruction when the exact short-time FT magnitude is known but also to applications in which the FT magnitude has been purposely or inadvertently modified. This occurs, for example, in speed-rate changes of speech for which the time scale of the short-time FT is purposely altered. In such cases, the resulting function of time and frequency is no longer a valid short-time FT. Nevertheless, reconstruction using the algorithms based on short-time Fourier magnitude alone provide a phase consistency in the reconstructed signal, which is highly desirable.

C. Reconstruction from Signed Fourier-Transform Magnitude

As was the case with FT phase, the high intelligibility of signed-magnitude-only synthesis suggests that, under some conditions, a signal may be uniquely specified by its signed FT magnitude. It was recently shown[21,22] that a real, finite-length, and causal (or anticausal) signal with no zeros on the unit circle is uniquely specified by its signed FT magnitude.[23] In addition, an iterative algorithm that alternately imposes the real, causal, and finite-length constraints in the time domain and the known signed magnitude in the frequency domain has been developed to reconstruct a signal from its signed FT magnitude. Although samples of the signed FT magnitude at a finite prespecified set of frequencies in general do not uniquely specify a signal, and the iterative algorithm uses the signed FT magnitude of the discrete Fourier transform (DFT), experimental results have shown that the algorithm reconstructs the correct sequence when a sufficiently large DFT size is chosen.

4. NEW RESULTS ON EXACT SIGNAL RECONSTRUCTION

In Section 3.C, we summarized previously reported results on exact reconstruction from signed FT magnitude. In Sections 4 we present some new generalizations and extensions of these results. Of principal importance is the fact that, as we demonstrate in Section 4.B, the restriction of finite extent previously imposed can be removed, and we present a somewhat more intuitive but formally correct justification than in previous publications. This result is based on the observation developed in Section 4.A that, since a finite-length sequence is uniquely specified to within a real scale factor by samples of its FT phase, an all-pass sequence is also. In Section 4.C we discuss new conditions, not previously reported to our knowledge, under which a one-dimensional signal can be reconstructed from only the sign information in Eq. (3) without the magnitude.

A. Reconstruction of an All-Pass Sequence from Phase Samples

A result to be used in Section 4.B and interesting in its own right is the fact that an Nth-order real, causal, stable, all-pass sequence can be uniquely recovered, to within a scale factor, from N samples of its phase in the interval $0 < \omega < \pi$. To develop this result, let $H_{ap}(z)$ denote the z transform of an Nth-order real, stable, all-pass sequence normalized in magnitude so that $H_{ap}(z)|_{z=1} = 1$. Then $H_{ap}(z)$ can be expressed in the form

$$H_{ap}(z) = z^{-N} \frac{\prod_{l=1}^{N} (1 - a_l z)}{\prod_{l=1}^{N} (1 - a_l^* z^{-1})}, \tag{11}$$

where $|a_l| < 1$. Define

$$R(z) = z^{-N} \prod_{l=1}^{N} (1 - a_l z) \tag{12}$$

and denote by $\theta_r(\omega)$ and by $\theta_{ap}(\omega)$ the unwrapped phase associated with $R(z)$ and $H_{ap}(z)$ on the unit circle. Then, since $(1 - a_l e^{j\omega})^* = (1 - a_l^* e^{-j\omega})$, $\theta_{ap}(\omega)$ can be expressed as

$$\theta_{ap}(\omega) = 2\theta_r(\omega) + N\omega$$

or

$$\theta_r(\omega) = \frac{1}{2} [\theta_{ap}(\omega) - N\omega]. \tag{13}$$

Since $R(z)$ is the z transform of a real, finite-length sequence that is zero outside the interval $[0, N]$ and has no zeros on the unit circle or in conjugate reciprocal pairs, then, from the results in Ref. 16, it is uniquely specified by N samples of $\theta_r(\omega)$ in the interval $0 < \omega < \pi$. Equivalently, then, from Eqs. (11) and (13), $H_{ap}(z)$ is uniquely specified by N samples of $\theta_{ap}(\omega)$ in the interval $0 < \omega < \pi$. This demonstration can be extended to include arbitrary all-pass sequences, but the result given for causal all-pass sequences is sufficient for the development that follows.

B. Reconstruction from Signed Fourier-Transform Magnitude

From the result in Section 4.A it follows that the results on signal reconstruction from signed FT magnitude discussed in Section 3.C and originally stated for finite-extent signals[21,22] generalize directly to the case of infinite-duration signals. In the following, we demonstrate[25] that a real, causal, and stable one-dimensional sequence (of either finite extent or infinite extent) with rational z transform and no zeros on the unit circle is uniquely specified by its signed FT magnitude.

Let $\alpha = \pi/2$ in Eq. (5), in which case $A_f^{\alpha}(\omega)$ is given by Eq. (4). From Eqs. (3) and (4), at points of discontinuity on the signed magnitude the phase must be either $\pi/2$ or $-(\pi/2)$. Specifically,

$$\theta_f(\omega) = \begin{cases} \dfrac{\pi}{2} & \text{if } A_f^{\alpha}(\omega_d) > 0 \\ -\dfrac{\pi}{2} & \text{otherwise} \end{cases} \tag{14}$$

where $A_f^{\alpha}(\omega)$ is discontinuous at ω_d. Since the original sequence is assumed to be causal and stable, all its poles are inside the unit circle. If N is the number of zeros outside the unit circle, from the argument principle,[24]

$$\hat{\theta}(2\pi) - \hat{\theta}(0) = -2\pi N, \tag{15}$$

where $\hat{\theta}(\omega)$ represents the unwrapped phase. Since $\hat{\theta}(\omega)$ is continuous, it must take on the values $\pm(\pi/2) \pm 2k\pi$ for at least $2N$ points on the interval $(0, 2\pi)$ or for at least N points on the interval $(0, \pi)$. Thus from the signed magnitude we can obtain the magnitude and N samples of the phase, where N is the number of zeros outside the unit circle. The value of N can also be determined from the samples of the unwrapped phase.

Next, we express $F(z)$, the z transform of $f(n)$, in the form

$$F(z) = A \frac{\prod_{l=1}^{N} (z^{-1} - a_l) \prod_{l=1}^{M} (1 - b_l z^{-1})}{\prod_{l-1}^{P} (1 - c_l z^{-1})}, \tag{16}$$

where $|a_l|$, $|b_l|$, and $|c_l|$ are all <1. Then $F(z)$ can be ex-

Table 1. Summary of Results on Fourier Synthesis from Partial Information

Signal Type	Intelligibility[a]
Magnitude-only signal	Not intelligible
Phase-only signal	Intelligible
Signed-magnitude-only signal	Intelligible
One-bit phase-only signal	Intelligible

[a] For both one-dimensional and multidimensional signals.

pressed as

$$F(z) = F_{mp}(z)H_{ap}(z), \tag{17}$$

where

$$F_{mp}(z) = A\frac{\prod_{l=1}^{N}(1 - a_l{}^*z^{-1})\prod_{l=1}^{M}(1 - b_l z^{-1})}{\prod_{l=1}^{P}(1 - c_l z^{-1})}$$

and

$$H_{ap}(z) = \frac{\prod_{l=1}^{N}(z^{-1} - a_l)}{\prod_{l=1}^{N}(1 - a_l{}^*z^{-1})}.$$

Here, $F_{mp}(z)$ denotes the z transform of the minimum-phase sequence associated with the magnitude $|F(\omega)|$, and $H_{ap}(z)$

Table 2. Summary of Results on Exact Reconstruction from Partial Information

Information	Constraints on Signal	Do Typical Sequences Satisfy Constraints?	Unique Specification within the Following Aspects	Unique Specification by Samples at a Finite Set of Arbitrary Frequencies?	Do Robust Algorithms Exist?
FT magnitude, multidimensional	Real, finite extent, nonfactorable z transform	Almost always	Translation, reflection with respect to origin, sign factor	Yes	Mixed results
FT magnitude, one-dimensional	Same	Few	Same	Yes	Yes
Short-time FT Magnitude, one-dimensional	Real, one-sided, number of consecutive zeros within sequence less than analysis window size	Almost always	Sign factor	Yes	Yes
FT Phase, multidimensional	Real, finite extent, no symmetric factors	Almost always	Positive scale factor	Yes	Yes
FT phase, one-dimensional	Real, finite extent, no zeros in conjugate reciprocal pairs or on the unit circle	Almost always	Positive scale factor	Yes	Yes
Signed FT Magnitude, multidimensional	Real, finite extent, one-sided, no zeros on a small region of the unit surface	Most	Unique	No[a]	Yes (based on limited results)
Signed FT magnitude, one-dimensional	Same as above with no zeros on the unit circle	Most	Unique	No	Yes (based on limited results)
Signed FT magnitude, one-dimensional	Real, stable, one-sided, rational z transform, no zeros on unit circle	Most	Unique	No	No
One-Bit FT phase, one-dimensional	Real, finite extent, either $x(n)$ or $x(-n)$ with all zeros inside unit circle and all poles at infinity	Few	Positive scale factor	No	No

[a] Note, however, that the conditions given for unique specification in terms of FT magnitude also apply for signed FT magnitude.

denotes the z transform of an all-pass sequence. Note that $H_{ap}(z)$ contains one pole and one zero (at conjugate reciprocal locations) for every zero of $F_{mp}(z)$ that must be reflected, that is, for every zero of $F(z)$ outside the unit circle. Since the magnitude of $F(\omega)$ uniquely specifies $F_{mp}(z)$ to within a sign factor that can be determined from $A_f(\omega)|_{\omega=0}$, the N phase samples of $F(z)$ specify N phase samples of $H_{ap}(z)$. Thus, from Section 4.A, $H_{ap}(z)$ is uniquely specified, and therefore $F(z)$ is uniquely specified.

The above discussion can be extended to include a more general definition of signed magnitude as given by Eq. (5). The only difference is that the signed magnitude now specifies the points where the phase crosses α or $\alpha - \pi$ instead of the points where the phase crosses $\pi/2$ or $-(\pi/2)$. This argument does not hold for $\alpha = 0$ since, in this special case, two points of discontinuity of the amplitude will occur at $\omega = 0$ and $\omega = \pi$, leaving only $N - 1$ points of discontinuity on the interval $(0, \pi)$.

The above discussion also applies, with simple modifications, to the case of a real, *anticausal*, and stable one-dimensional signal with rational z transform and with no zeros on the unit circle.

C. Reconstruction from One-Bit Fourier-Transform Phase

Under certain conditions, a one-dimensional signal can be reconstructed exactly only from $S_f{}^\alpha(\omega)$, the one-bit FT phase information defined in Eq. (8) without the magnitude $M_f(\omega)$, as was required in Section 4.B. This is the case if the sequence is of finite length, is causal, and has all its zeros outside the unit circle. For such a sequence, $S_f{}^\alpha(\omega)$ will change sign over $0 < \omega < \pi$ at least as many times as the total number of zeros of the sequence. Since $\theta_f(\omega)$ can be exactly determined from Eq. (14) at the frequencies at which $S_x{}^\alpha(\omega)$ changes its sign, the sequence can be uniquely determined within a scale factor from the one-bit FT phase information. A similar result also holds for the case in which the one-dimensional sequence is of finite length, is anticausal, and has all its zeros inside the unit circle.

Whereas most signals found in practice would not satisfy either of these conditions, any signal meeting the appropriate causality constraints can be made to do so by adding a sufficiently large impulse at the beginning or the end of the signal. Thus a signal could in theory be reconstructed by first adding an impulse at the beginning or at the end, advancing or delaying it to make it anticausal or causal, taking the sign of the real part of the FT of this new sequence, reconstructing the new sequence from the sign of the real part, and then removing the impulse. As was discussed in Section 2.B, synthesis from $S_f{}^\alpha(\omega)$ alone has reasonably high intelligibility.

5. SUMMARY

We have reviewed a number of previous results and presented some new results on the Fourier synthesis from partial Fourier-domain information and exact reconstruction from partial Fourier-domain information. Tables 1 and 2 summarize the results discussed in this paper. Table 1 summarizes results of the Fourier synthesis from partial information, and Table 2 summarizes results of the exact reconstruction from partial information.

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23. The bipolar function $S_f{}^\alpha(\omega)$ is defined in this paper by Eq. (8), as shown below:

$$S_f{}^\alpha(\omega) = 1 \quad \text{for } \alpha - \pi < \theta_f(\omega) \le \alpha. \qquad \text{(R1)}$$
$$= -1 \quad \text{otherwise}$$

The definition used in Refs. 21 and 22 for $S_f{}^\alpha(\omega)$ is slightly different from Eq. (R1), as shown below:

$$S_f{}^\alpha(\omega) = 1 \quad \text{for } \alpha - \pi \le \theta_f(\omega) \le \alpha. \qquad \text{(R2)}$$
$$= -1 \quad \text{otherwise}$$

Since the $S_f{}^\alpha 8\omega)$ in Eq. (R2) can be derived from that in Eq. (R1), the results in Refs. 21 and 22 also apply when $S_f{}^\alpha(\omega)$ is defined by Eq. (R1).

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25. The demonstration applies to the definition of $S_f{}^\alpha(\omega)$ given by Eq. (8) and does not apply to the definition of $S_f{}^\alpha(\omega)$ used in Refs. 21 and 22. For the definition of $S_f{}^\alpha(\omega)$ in Refs. 21 and 22, see Ref. 23.