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Reconstruction of complex-valued propagating wave fields using the Hilbert–Hankel transform

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In this paper the theory of the unilateral inverse Fourier transform and the unilateral Hankel transform is developed. The consistency between each transform and its bilateral version leads to an approximate real-part sufficiency condition for complex-valued one-dimensional even signals and two-dimensional circularly symmetric signals. The two-dimensional result is used in a reconstruction algorithm that is applied to synthetic and experimental underwater acoustic fields.

1. INTRODUCTION

A well-known property in Fourier-transform theory is that causality in one domain implies real-part sufficiency in the alternate domain. This property is the basis for the fact that the real and imaginary components of a signal are related by the Hilbert transform, if the spectrum of the signal is causal. In wave propagation problems, it is the circularly symmetric two-dimensional Fourier transform or, equivalently, the Hankel transform that is of central importance. Because of the symmetry in such problems, the condition of causality is not applicable. In one dimension, the counterpart of the circularly symmetric signal is the even signal. The one-dimensional Fourier transform of an even signal is also even, and thus, again, the condition of causality is not applicable.

In our work we have shown that under some conditions it is possible to relate approximately the real and imaginary components of a one-dimensional even signal or a two-dimensional circularly symmetric signal. The approximation is based on the validity of the unilateral inverse Fourier transform in one dimension and that of the unilateral Hankel transform, referred to as the Hilbert–Hankel transform, in two dimensions. In this paper we develop the approximate real-part sufficiency condition in one and two dimensions by using these transforms. The two-dimensional result forms the basis for a reconstruction algorithm in which the real (or imaginary) component of a complex-valued propagating wave field is obtained from the imaginary (or real) component. The algorithm is illustrated with synthetic and experimental underwater acoustic fields.

In Section 2 first a review of one-dimensional exact analytic signals is provided, and then the theory of one-dimensional approximate analytic signals is presented. A number of statements involving the unilateral Fourier transform, the unilateral inverse Fourier transform, causality, and approximate real-part–imaginary-part sufficiency are made. In Section 3 the theory is extended to two-dimensional circularly symmetric signals. These signals, which can be equivalently described in terms of the Hankel transform, are directly related to two-dimensional fields propagating in horizontally stratified media. We will show that, under some conditions, it is possible to relate approximately the real and imaginary components of these fields. To do this, a unilateral version of the Hankel transform, referred to as the Hilbert–Hankel transform, is described. The transform can be used to synthesize approximately an outgoing wave field, and its consistency with the Hankel transform will be shown to imply an approximate relationship between the real and imaginary components of the field. The properties of the Hilbert–Hankel transform and the relationship of this transform to several other transforms will be discussed. In Section 4 an asymptotic version of the Hilbert–Hankel transform is developed. This transform leads to an efficient algorithm for reconstructing a complex-valued acoustic field from its real or imaginary part. The algorithm is applied to both synthetic and experimental underwater acoustic fields.

2. ONE-DIMENSIONAL THEORY: THE UNILATERAL INVERSE FOURIER TRANSFORM

From the theory of analytic functions, a complex-valued function of a complex-valued variable is analytic at a point if it is both single valued and uniquely differentiable at that point. A necessary condition for a unique derivative is that the real and imaginary components of the function satisfy the Cauchy–Riemann conditions,[1] which involve the partial derivatives of the function. If these partial derivatives are also continuous, the Cauchy–Riemann conditions form a necessary and sufficient condition for analyticity at a given point.

The Cauchy–Riemann conditions imply that the real and imaginary components of a function cannot be chosen independently, if the function is to be analytic. These conditions imply that if the real (or imaginary) component is specified within the region of analyticity, the imaginary (or real) component can be determined. In some cases, knowledge of one of the components along only the boundary of the analytic region is sufficient to determine the alternate component.[2,3] For example, for the region of analyticity defined by a circle centered at the origin of the complex plane, integral relationships between the real and imaginary components of the function referred to as Poisson integrals have been developed. Similarly, for the region of analyticity defined by a half-plane that includes the real or imaginary axis, integral relationships between the real and imaginary components of the function along the axis, referred to as Hilbert transform integrals, have been developed.

In a signal processing context, there is a related concept that states that a one-sided, or causal, condition in one domain implies a real-part–imaginary-part sufficiency condition in the alternate domain; that is, a complex-valued signal has a real-part–imaginary-part sufficiency condition if its Fourier transform is causal and vice versa. A signal that can be exactly synthesized in terms of a one-sided Fourier transform is often referred to as an analytic signal.

To summarize the relationships, let us consider a complex-valued signal $f(t)$ with a Fourier transform consisting of only positive components, i.e.,

$$f(t) = \frac{1}{2\pi}\int_0^\infty F(\omega)e^{j\omega t}\mathrm{d}\omega. \tag{1}$$

It can be shown that the one-sided integral in Eq. (1) implies that $f(t)$ is an analytic function in the upper half of the complex t plane.[4,5] This one-sided condition connects the theory of the analytic signal with the theory of the analytic function.

Let us express $f(t)$ in terms of its real and imaginary components, for t real, as

$$f(t) = g(t) + j\hat{g}(t), \tag{2}$$

with $G(\omega)$ and $\hat{G}(\omega)$ denoting the Fourier transforms of $g(t)$ and $\hat{g}(t)$, respectively. The relationships between the real and imaginary components of $f(t)$ and their Fourier transforms can then be summarized as follows:

$$F(\omega) = 2G(\omega)u(\omega),$$
$$\hat{G}(\omega) = -j\ \mathrm{sgn}(\omega)G(\omega), \tag{3}$$

where $u(\cdot)$ is the unit step function. The signals $g(t)$ and $\hat{g}(t)$ form a *Hilbert transform pair.*

Although the Hilbert transform relationship between $g(t)$ and $\hat{g}(t)$ is conveniently summarized in the frequency domain, it is also possible to use the convolution property of Fourier transforms to develop the relationship in the time domain. Specifically, using the inverse Fourier transform of $-j\ \mathrm{sgn}(\omega)$, it can be shown that

$$\hat{g}(t) = \frac{1}{\pi t} * g(t) \tag{4}$$

and similarly that

$$g(t) = -\frac{1}{\pi t} * \hat{g}(t), \tag{5}$$

where the integrals are interpreted as Cauchy principal valued.

It is possible to extend some of the properties of analytic signals to signals that do not possess a one-sided Fourier transform. Our primary interest has been in two-dimensional circularly symmetric signals, which are related to the two-dimensional circularly symmetric Fourier transform or, equivalently, to the Hankel transform. However, the extension of the theory of analytic functions can best be presented by first considering the one-dimensional case. In the remainder of this section the theory of one-dimensional signals that are approximately analytic is developed.

We now consider a complex-valued signal $f(t)$ for which the Fourier transform exists and which has the Laplace transform $F_0(s)$ given by

$$F_0(s) = \int_{-\infty}^\infty f(t)e^{-st}\mathrm{d}t. \tag{6}$$

In general, the Fourier transform $F(\omega)$ is a two-sided function of ω, and thus the signal $f(t)$ is represented in terms of the inverse Fourier transform as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{j\omega t}\mathrm{d}\omega. \tag{7}$$

We also define the related signal $f_u(t)$ in terms of the unilateral inverse Fourier transform as

$$f_u(t) \equiv \frac{1}{2\pi}\int_0^\infty F(\omega)e^{j\omega t}\mathrm{d}\omega. \tag{8}$$

Note that $f_u(t)$ is an analytic signal, since its Fourier transform is causal.

One way of extending the theory of analytic signals to the signal $f(t)$, which has a two-sided Fourier transform, is to require that

$$f(t) \sim f_u(t). \tag{9}$$

Thus a signal $f(t)$ that can be approximated by a unilateral version of its inverse Fourier transform can be considered approximately analytic.

The condition that a signal can be accurately synthesized in terms of its unilateral inverse Fourier transform is rather restrictive, and it certainly does not apply to any arbitrary signal. For example, consider a signal, composed of a sum of complex exponentials, that has a rational Laplace transform. In Fig. 1 we have indicated the positions of several poles in the s plane, corresponding to the arbitrarily chosen signal, as well as the region of convergence for the Laplace transform. The condition that $f(t) \sim f_u(t)$ is equivalent to the statement that the inverse Laplace transform contour $C_1 + C_2$ can be approximately replaced by the contour C_1. The approximation will be poor if a pole, such as P_3, is located in quadrant III or IV of the s plane. Essentially, the effects of this pole, quite important in determining the character of the corresponding signal $f(t)$, are only negligibly included by integrating along the positive imaginary axis only. That is, if $f(t)$ is exactly synthesized as

$$f(t) = \frac{1}{2\pi j}\int_{C_1+C_2} F_0(s)e^{st}\mathrm{d}s, \tag{10}$$

so that

$$f(t) = \frac{1}{2\pi j}\int_{C_1} F_0(s)e^{st}\mathrm{d}s + \frac{1}{2\pi j}\int_{C_2} F_0(s)e^{st}\mathrm{d}s, \tag{11}$$

the pole at position P_3 contributes primarily to the second of these integrals. Thus the approximation

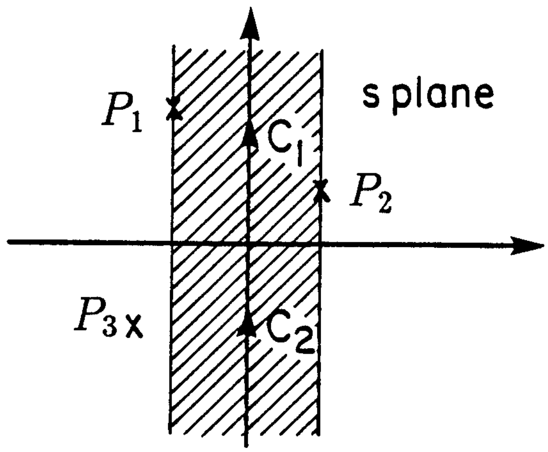

Fig. 1. Complex s plane, indicating positions of poles, the inverse Laplace transform integration contour, and the Laplace transform region of convergence.

$$f(t) \sim \frac{1}{2\pi j}\int_{C_1} F_0(s)e^{st}\mathrm{d}s = \frac{1}{2\pi}\int_0^{\infty} F(\omega)e^{j\omega t}\mathrm{d}\omega \qquad (12)$$

is not accurate because of the position of pole P_3 in the s plane. However, if there are no poles in quadrant III or IV of the s plane, the unilateral inverse transform can yield an accurate version of $f(t)$. This can be argued on the basis of the fact that poles such as P_1 and P_2 contribute primarily to the contour integral C_1 and only negligibly to the contour integral C_2.

This concept of approximate analyticity is intuitive—if the signal $f(t)$ can be approximated by the analytic signal $f_u(t)$, $f(t)$ is approximately analytic. Although it is possible to develop an approximate relationship between the real and imaginary components of $f(t)$, there are no further consequences of the relationship $f(t) \sim f_u(t)$. However, as will be discussed in the remainder of this section, some interesting relationships result if additional restrictions are placed on the signal $f(t)$ and the definition of approximate analyticity is slightly modified. For example, there are interesting consequences that arise if $f(t)$ is restricted to be a causal signal or an even signal. Essentially, by considering $f(t)$ to be an even signal, the case of the causal signal can be treated as well, since an even (or causal) signal can always be invertibly constructed from a causal (or even) signal.

To develop the theory, let us now consider the complex-valued signal $f(t)$ to be even. We denote the Laplace transform by $F_0(s)$ and again assume that the Fourier transform $F(\omega) = F_0(s)|_{s=j\omega}$ exists. Since the signal is even, it follows that its Fourier transform and Laplace transform must also be even. We will find it convenient to define not only the Fourier transform and the inverse Fourier transform but their unilateral counterparts as well. Thus

$$f(t) \equiv \mathcal{F}^{-1}\{F(\omega)\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\mathrm{d}\omega, \qquad (13)$$

$$f_u(t) \equiv \mathcal{F}_u^{-1}\{F(\omega)\} \equiv \frac{1}{2\pi}\int_0^{\infty} F(\omega)e^{j\omega t}\mathrm{d}\omega, \qquad (14)$$

$$F(\omega) \equiv \mathcal{F}\{f(t)\} \equiv \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{d}t, \qquad (15)$$

$$F_u(\omega) \equiv \mathcal{F}_u\{f(t)\} \equiv \int_0^{\infty} f(t)e^{-j\omega t}\mathrm{d}t, \qquad (16)$$

where $f_u(t)$ represents the unilateral inverse Fourier transform of $F(\omega)$, and $F_u(\omega)$ represents the unilateral Fourier transform of $f(t)$. Symbolically, $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier transform and the inverse Fourier transform operations and $\mathcal{F}_u$ and $\mathcal{F}_u^{-1}$ represent the unilateral Fourier transform and the unilateral inverse Fourier transform operations.

Note that, although $\mathcal{F}$ and $\mathcal{F}^{-1}$ are inverse operations, $\mathcal{F}_u$ and $\mathcal{F}_u^{-1}$ are not necessarily inverse operations. Additionally, it is recognized that both $f_u(t)$ and $F_u(\omega)$ are analytic signals, since their Fourier transforms are causal. Therefore the real and imaginary components of $f_u(t)$ are exactly related by the Hilbert transform, and the real and imaginary components of $F_u(\omega)$ are exactly related by the Hilbert transform. Also note that, since $f(t)$ and thus $F(\omega)$ are even signals, they can be synthesized in terms of the cosine transform as

$$f(t) = \frac{1}{\pi}\int_0^{\infty} F(\omega)\cos \omega t\mathrm{d}\omega, \qquad (17)$$

$$F(\omega) = 2\int_0^{\infty} f(t)\cos \omega t\mathrm{d}t. \qquad (18)$$

To extend the theory of analytic signals to the signal $f(t)$, we will require that $f(t)$ satisfy the condition

$$f(t) \sim f_u(t), \qquad t > 0; \qquad (19)$$

that is, only those functions $f(t)$ that can be approximated by the unilateral inverse Fourier transform for positive values of t will be considered. Note that this condition differs from the condition in relation (9). Specifically, the unilateral inverse Fourier transform is required to synthesize the even signal $f(t)$ only for positive values of t. The even function $f(t)$ will be defined as approximately analytic if the condition in relation (19) is satisfied. To the extent that the approximation in relation (19) is valid, there will also exist an approximate relationship between the real and imaginary components of $f(t)$, for $t > 0$. This result is the basis for a number of statements that will now be made.

If $f(t) \sim f_u(t)$ for $t > 0$, then:

Statement 2.1 *The real and imaginary components of $f(t)$ must be approximately related by the Hilbert transform for $t > 0$.*

Statement 2.2 *The unilateral inverse Fourier transform $f_u(t)$ is approximately causal.*

Statement 2.3 *The unilateral Fourier transform and unilateral inverse Fourier transform are related by the approximation*

$$\mathcal{F}_u\{\mathcal{F}_u^{-1}\{F(\omega)\}\} \sim F(\omega)u(\omega). \qquad (20)$$

Statement 2.4 *The real and imaginary components of $F(\omega)$ must be approximately related by the Hilbert transform for $\omega > 0$.*

Statement 2.1 follows as a consequence of the fact that the real and imaginary components of $f_u(t)$ are related exactly by the Hilbert transform.

To justify statement 2.2, we note that $f(t) \sim f_u(t)$, $t > 0$ implies that

$$\frac{1}{2\pi}\int_{-\infty}^{0} F(\omega)e^{j\omega t}\mathrm{d}\omega \sim 0, \qquad t > 0, \qquad (21)$$

so that

$$\frac{1}{2\pi}\int_0^\infty F(\omega)e^{-j\omega t}\mathrm{d}\omega \sim 0, \qquad t > 0. \tag{22}$$

The latter step follows from the fact that $F(\omega)$ is even in ω. From relation (22) and the definition of the unilateral inverse Fourier transform, it can be seen that

$$f_u(-t) \sim 0, \qquad t > 0, \tag{23}$$

and thus

$$f_u(t) \sim 0, \qquad t < 0. \tag{24}$$

Therefore, under the condition that $f(t) \sim f_u(t)$ for $t > 0$, the unilateral inverse Fourier transform must be approximately causal.

Let us next consider statement 2.3. As pointed out earlier, the unilateral Fourier transform and the unilateral inverse Fourier transform are generally not inverse operations. However, from the definition of $f_u(t)$,

$$\mathcal{F}_u\{\mathcal{F}_u^{-1}\{F(\omega)\}\} = \mathcal{F}_u\{f_u(t)\}. \tag{25}$$

Since $f_u(t)$ is approximately causal, it follows that

$$\mathcal{F}_u\{f_u(t)\} \sim \mathcal{F}\{f_u(t)\}. \tag{26}$$

Additionally, from the definition of $f_u(t)$, it is also apparent that

$$\mathcal{F}\{f_u(t)\} = F(\omega)u(\omega). \tag{27}$$

When the three relations are combined, statement 2.3 follows.

To justify statement 2.4, we use statement 2.3 to establish that

$$\mathcal{F}_u\{F_u^{-1}\{F(\omega)\}\} \sim F(\omega)u(\omega), \tag{28}$$

so that

$$\mathcal{F}_u\{f_u(t)\} \sim F(\omega)u(\omega). \tag{29}$$

The signal $F(\omega)u(\omega)$ is approximately analytic, since it is related to the one-sided Fourier transform on the left-hand side of relation (29). Thus, since $F(\omega)u(\omega)$ is approximately analytic, its real and imaginary parts must be related by the Hilbert transform.

Although the specific statements made involve the relationships among the Fourier transform, the inverse Fourier transform, and their unilateral counterparts, it is also possible to derive a number of interesting relationships between the cosine and sine transforms that compose these. Several of these relationships are derived in Appendix A.

To summarize briefly, in this section we have reviewed the one-dimensional theory of exact and approximate analytic signals. The theory of an exact analytic signal was presented in terms of the properties of the Fourier transform. This theory was then extended to develop the notion of a signal that is approximately analytic. Although such a signal does not have a causal Fourier transform, its real and imaginary parts can be approximately related by the Hilbert transform. The necessary condition for an even signal to possess this property is that its causal portion must be accurately synthesized by the unilateral inverse Fourier transform. This condition implies a number of other interesting consequences, including an approximate real-part–imaginary-part sufficiency for both the causal part of the signal and the causal part of its Fourier transform and an inverse relationship between the unilateral Fourier transform and the unilateral inverse Fourier transform.

3. TWO-DIMENSIONAL THEORY: THE HILBERT-HANKEL TRANSFORM

In the previous section, we considered conditions under which a one-dimensional complex-valued signal is either exactly or approximately analytic. In this section, the theory of approximate analyticity will be extended to two-dimensional circularly symmetric signals. Although the theory can be developed for the more general multidimensional case by considering the multidimensional version of the unilateral inverse Fourier transform, we will focus primarily on the special case of a two-dimensional circularly symmetric signal because of its practical relevance to a wave field propagating in a horizontally stratified medium.

Consider a two-dimensional signal $p(x, y)$ and its two-dimensional Fourier transform $\tilde{g}(k_x, k_y)$, which are related by

$$p(x, y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{g}(k_x, k_y)\exp[j(k_x x + k_y y)]\mathrm{d}k_x\mathrm{d}k_y. \tag{30}$$

If $p(x, y)$ is circularly symmetric, then $\tilde{g}(k_x, k_y)$ is also circularly symmetric, and the relationship in Eq. (30) can be written as

$$p(r) = \int_0^\infty g(k_r)J_0(k_r r)k_r\mathrm{d}k_r, \tag{31}$$

where $r = (x^2 + y^2)^{1/2}$, $k_r = (k_x + k_y)^{1/2}$, $g(k_r) = \tilde{g}(k_r)/(2\pi)$, and $J_0(\cdot)$ is the zero-order Bessel function.

The relationship described in Eq. (31) is a Hankel transform. Note that $p(r)$ may be a complex-valued signal, but, because of the circular symmetry involved, the condition of causality is not applicable, and the real and imaginary components of $p(r)$ are generally unrelated. The Hankel transform is self-inverse, so that

$$g(k_r) = \int_0^\infty p(r)J_0(k_r r)r\mathrm{d}r. \tag{32}$$

The two-dimensional circularly symmetric signal $p(r)$ is analogous to the one-dimensional even signal $f(t)$ considered in the previous section.

To develop the property of approximate analyticity for two-dimensional circularly symmetric signals, we must develop a unilateral version of the Hankel transform. That is, with analogy to the one-dimensional bilateral inverse Fourier transform and one-dimensional unilateral inverse Fourier transform, we wish to develop the Hankel transform and its unilateral version. In examining Eq. (31) we see that in one sense the Hankel transform is already unilateral, since the limits of integration are from zero to infinity. However, this version of the Hankel transform is actually analogous to the one-sided cosine transform for one-dimensional signals. Here, we want to develop the transform analogous to the Fourier transform and then consider its unilateral counterpart.

To this end, the zero-order Bessel function of the first kind is written in terms of Hankel functions[6] as

$$J_0(k_r r) = \tfrac{1}{2}[H_0^{(1)}(k_r r) + H_0^{(2)}(k_r r)], \quad (33)$$

so that Eq. (31) becomes

$$p(r) = \tfrac{1}{2}\int_0^\infty g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r + \tfrac{1}{2}\int_0^\infty g(k_r)H_0^{(2)}(k_r r)k_r \mathrm{d}k_r, \quad (34)$$

which is valid for both positive and negative values of r. The expression can be simplified by using the property[7] that

$$H_0^{(2)}(ze^{-j\pi}) = -H_0^{(1)}(z) \quad (35)$$

to yield

$$p(r) = \tfrac{1}{2}\int_{-\infty}^\infty g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r, \qquad r > 0. \quad (36)$$

The signal $p(r)$ can be determined for negative values of r by utilizing this equation and the fact that $p(r) = p(-r)$. However, it is important to recognize that the bilateral transform in Eq. (36) correctly synthesizes the signal $p(r)$ for positive values of r only. Specifically, although $p(r)$ is an even function of r, the Hankel function $H_0^{(1)}(k_r r)$ is not an even function of r, nor is $g(k_r)$ an odd function of k_r, and thus the expression in Eq. (36) is not correct for $r < 0$. The correct expression for $r < 0$ can be obtained, by using properties of $H_0^{(1)}(k_r r)$ and $H_0^{(2)}(k_r r)$, as

$$p(r) = \tfrac{1}{2}\int_{-\infty}^\infty g(k_r)H_0^{(2)}(k_r r)k_r \mathrm{d}k_r, \qquad r < 0. \quad (37)$$

Alternatively, a bilateral expression that describes $p(r)$ correctly for both positive and negative values of r can be written as

$$p(r) \equiv \mathcal{H}^{-1}\{g(k_r)\} \equiv \tfrac{1}{2}\int_{-\infty}^\infty g(k_r)H_0^{(1)}(k_r|r|)k_r \mathrm{d}k_r. \quad (38)$$

The transform in Eq. (38) will be referred to as the bilateral inverse Hankel transform.

The unilateral version of the transform in Eq. (38) is defined as

$$p_u(r) \equiv \mathcal{H}_u^{-1}\{g(k_r)\} \equiv \tfrac{1}{2}\int_0^\infty g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r. \quad (39)$$

The operator $\mathcal{H}_u^{-1}$ is appropriately thought of as the unilateral inverse Hankel transform. However, because the Hankel transform in Eq. (31) is already unilateral, the name unilateral Hankel transform is ambiguous. Instead, the transform defined in Eq. (39) will be referred to as the Hilbert–Hankel transform because of its close relationship to the Hankel transform and because, as will be discussed shortly, the transform implies an approximate relationship between the real and imaginary components of $p(r)$.

The bilateral inverse Hankel transform and the Hilbert–Hankel transform can be written in alternate forms by utilizing the relationship

$$H_0^{(1)}(k_r r) = J_0(k_r r) + jY_0(k_r r), \quad (40)$$

where $Y_0(k_r r)$ is the zero-order Bessel function of the second kind, also referred to as the Neumann function.[6] We note that both $J_0(k_r r)$ and $Y_0(k_r r)$ are real-valued functions for real-valued arguments. By using this relationship, the bilateral inverse Hankel transform can be written as

$$\mathcal{H}^{-1}\{g(k_r)\} \equiv \tfrac{1}{2}\int_{-\infty}^\infty g(k_r)[J_0(k_r r) + jY_0(k_r|r|)]k_r \mathrm{d}k_r, \quad (41)$$

and the Hilbert–Hankel transform can be written as

$$\mathcal{H}_u^{-1}\{(k_r)\} \equiv \tfrac{1}{2}\int_0^\infty g(k_r)[J_0(k_r r) + jY_0(k_r r)]k_r \mathrm{d}k_r. \quad (42)$$

It is also possible to develop a bilateral transform for the inverse relationship between $p(r)$ and $g(k_r)$. From Eq. (32) and the relationship in Eq. (35), it follows that

$$g(k_r) \equiv \mathcal{H}\{p(r)\} \equiv \tfrac{1}{2}\int_{-\infty}^\infty p(r)H_0^{(1)}(|k_r|r)r\, \mathrm{d}r. \quad (43)$$

This expression will be referred to as the bilateral Hankel transform. Note that the bilateral Hankel transform and the bilateral inverse Hankel transform are identical operators, although they apply to different domains.

The most obvious definition for the unilateral version of this tranform is obtained by replacing the lower limit in Eq. (43) by zero. However, we will find it convenient to define the transform differently. In particular, the unilateral version of the transform will be defined as

$$g_u(k_r) \equiv \mathcal{H}_u\{p(r)\} \equiv \tfrac{1}{2}\int_0^\infty p(r)[J_0(k_r r) - j\mathbf{H}_0(k_r r)]r\, \mathrm{d}r. \quad (44)$$

The function $\mathbf{H}_0(z)$ is the zero-order Struve function.[7,8] The zero-order Struve function and the zero-order Bessel function of the first kind, $J_0(z)$, form a Hilbert transform pair.[5] To demonstrate this, $J_0(z)$ is expressed in terms of a Fourier synthesis integral as

$$J_0(z) = \frac{1}{2\pi}\int_{-1}^{1} \frac{2}{(1-\omega^2)^{1/2}} e^{j\omega z}\mathrm{d}\omega. \quad (45)$$

To compute the Hilbert transform $\hat{J}_0(z)$, the integrand in this expression is multiplied by $-j\,\mathrm{sgn}(\omega)$ to yield

$$\hat{J}_0(z) = \frac{1}{2\pi}\int_{-1}^{1} -j\,\mathrm{sgn}(\omega)\frac{2}{(1-\omega^2)^{1/2}} e^{j\omega z}\mathrm{d}\omega$$

$$= \frac{2}{\pi}\int_0^1 \frac{\sin \omega z}{(1-\omega^2)^{1/2}}\mathrm{d}\omega. \quad (46)$$

The last integral is also the integral representation for the zero-order Struve function.[7]

The transform defined in Eq. (44) has also been considered by Papoulis[5,9] and has been referred to as the complex Hankel transform. It is noted that $g_u(k_r)$ must be an analytic signal, since its real and imaginary components are related by the Hilbert transform. From preceding discussions, this implies that the one-dimensional Fourier transform of $g_u(k_r)$ must be causal. This fact and the use of the projection slice theorem for two-dimensional Fourier transforms[10,11] provide the basis for expressing the complex Hankel transform in terms of the Abel transform, which is the projection of the two-dimensional circularly symmetric function and the one-dimensional Fourier transform. Specifically, the complex Hankel transform of $p(r)$ can be determined by computing the Abel transform of $p(r)$, retaining the causal portion, and computing the one-dimensional Fourier transform. The relationships among the Abel, Fourier, Hankel, and complex Hankel transforms are summarized in Fig. 2. In this figure, the operator $\mathcal{A}$ refers to the Abel transform, defined as

$$p_A(x) = \mathcal{A}\{p(r)\} \equiv \frac{1}{2\pi}\int_{-\infty}^\infty p(r)\mathrm{d}y, \quad (47)$$

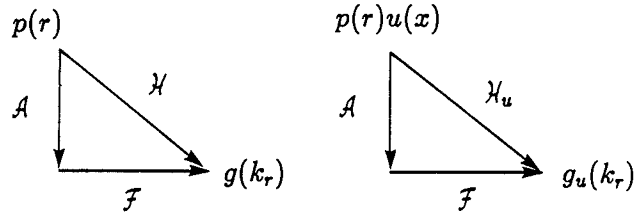

Fig. 2. Relationships among the Abel, Fourier, Hankel, and complex Hankel transforms.

where $r = (x^2 + y^2)^{1/2}$. We note that

$$\mathcal{A}\{p(r)u(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(r)u(x)\mathrm{d}y = p_A(x)u(x). \tag{48}$$

The definitions for the bilateral and unilateral transforms are now summarized:

$$p(r) \equiv \mathcal{H}^{-1}\{g(k_r)\} \equiv 1/2\int_{-\infty}^{\infty} g(k_r)[J_0(k_r r) + jY_0(k_r|r|)]k_r\mathrm{d}k_r, \tag{49}$$

$$p_u(r) \equiv \mathcal{H}_u^{-1}\{g(k_r)\} \equiv 1/2\int_{0}^{\infty} g(k_r)[J_0(k_r r) + jY_0(k_r r)]k_r\mathrm{d}k_r, \tag{50}$$

$$g(k_r) \equiv \mathcal{H}\{p(r)\} \equiv 1/2\int_{-\infty}^{\infty} p(r)[J_0(k_r r) + jY_0(|k_r|r)]r\,\mathrm{d}r, \tag{51}$$

$$g_u(k_r) \equiv \mathcal{H}_u\{p(r)\} \equiv 1/2\int_{0}^{\infty} p(r)[J_0(k_r r) - j\mathbf{H}_0(k_r r)]r\,\mathrm{d}r, \tag{52}$$

where $p_u(r)$ represents the Hilbert–Hankel transform of $g(k_r)$ and $g_u(k_r)$ represents the complex Hankel transform of $p(r)$. These equations are analogous to Eqs. (13)–(16), which were developed in the one-dimensional context in the preceding section. It is also convenient to define the following transforms and symbolic notation:

$$J_0\{g(k_r)\} \equiv \int_0^{\infty} g(k_r)J_0(k_r r)k_r\mathrm{d}k_r, \tag{53}$$

$$Y_0\{g(k_r)\} \equiv \int_0^{\infty} g(k_r)Y_0(k_r r)k_r\mathrm{d}k_r, \tag{54}$$

$$\mathbf{H}_0\{g(k_r)\} \equiv \int_0^{\infty} g(k_r)\mathbf{H}_0(k_r r)k_r\mathrm{d}k_r. \tag{55}$$

To develop the theory of two-dimensional circularly symmetric signals that are approximately analytic, we will require that

$$p(r) \sim p_u(r), \qquad r > 0; \tag{56}$$

that is, only circularly symmetric signals $p(r)$ that can be approximated by the Hilbert–Hankel transform for $r > 0$ will be considered approximately analytic. This condition is analogous to the condition stated in the previous section, that $f(t) \sim f_u(t)$, $t > 0$. To the extent that the approximation in relation (56) is valid, there will also exist an approximate relationship between the real and imaginary components of $p(r)$, for $r > 0$. This result is the basis for the first of several statements that will now be discussed. The statements will closely parallel the one-dimensional versions in the preceding section.

If $p(r) \sim p_u(r)$ for $r > 0$, then:

Statement 3.1 *The real and imaginary components of $p(r)$ must be approximately related for $r > 0$ by*

$$\mathrm{Re}[p(r)] \sim -Y_0\{J_0\{\mathrm{Im}[p(r)]\}\},$$
$$\mathrm{Im}[p(r)] \sim Y_0\{J_0\{\mathrm{Re}[p(r)]\}\}. \tag{57}$$

Statement 3.2 *The Hilbert–Hankel transform $p_u(r)$ is approximately causal.*

Statement 3.3 *The Hilbert–Hankel transform and the complex Hankel transform are related by the approximation*

$$\mathcal{H}_u\{\mathcal{H}_u^{-1}\{g(k_r)\}\} \sim g(k_r)u(k_r). \tag{58}$$

Statement 3.4 *The real and imaginary components of $g(k_r)$ must be approximately related by the Hilbert transform for $k_r > 0$.*

Statement 3.1 can be justified as follows. The condition that $p(r) \sim p_u(r)$, $r > 0$ can be equivalently written as

$$J_0\{g(k_r)\} \sim \mathcal{H}_u^{-1}\{g(k_r)\}, \qquad r > 0, \tag{59}$$

so that

$$J_0\{g(k_r)\} \sim 1/2 J_0\{g(k_r)\} + j1/2 Y_0\{g(k_r)\}, \qquad r > 0. \tag{60}$$

By using the facts that J_0 and Y_0 are real operators and that $\mathrm{Re}[p(r)] = J_0\{\mathrm{Re}[g(k_r)]\}$, $\mathrm{Im}[p(r)] = J_0\{\mathrm{Im}[g(k_r)]\}$, the statement is established by equating real and imaginary parts on both sides of relation (60).

Statement 3.2 will now be justified. Note that, from Eq. (50), the Hilbert–Hankel transform is defined for all values of r. Thus the causality condition in statement 3.2 is not a consequence of the definition of the Hilbert–Hankel transform but rather is a consequence of the condition that $p(r) \sim p_u(r)$, $r > 0$. We note that $p(r) \sim p_u(r)$, $r > 0$ implies that

$$1/2\int_{-\infty}^{0} g(k_r)\,H_0^{(1)}(k_r r)k_r\mathrm{d}k_r \sim 0, \qquad r > 0, \tag{61}$$

so that

$$-1/2\int_{0}^{\infty} g(k_r)H_0^{(1)}(-k_r r)k_r\mathrm{d}k_r \sim 0, \qquad r > 0. \tag{62}$$

The latter step follows from the fact that $g(k_r)$ is even in k_r. From this equation and the definition of the Hilbert–Hankel transform, it can be seen that

$$p_u(-r) \sim 0, \qquad r > 0, \tag{63}$$

and thus

$$p_u(r) \sim 0, \qquad r < 0. \tag{64}$$

Therefore, under the condition that $p(r) \sim p_u(r)$, $r > 0$, the Hilbert–Hankel transform must be approximately causal.

Let us next consider statement 3.3. The complex Hankel transform can be written in terms of the operators J_0, Y_0, and $\mathbf{H}_0$ as

$$\mathcal{H}_u\{p_u(r)\} = 1/4 g(k_r) + 1/4\mathbf{H}_0\{Y_0\{g(k_r)\}\} + 1/4 j[J_0\{Y_0\{g(k_r)\}\} - \mathbf{H}_0\{J_0\{g(k_r)\}\}], \tag{65}$$

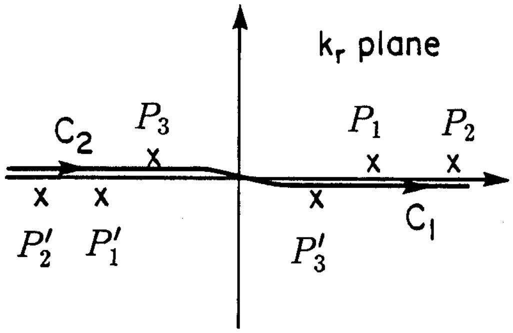

Fig. 3. Complex k_r plane, indicating positions of poles and the bilateral inverse Hankel-transform contour.

which is valid for all k_r. From the approximation $p(r) \sim p_u(r)$, $r > 0$, it then follows that

$$J_0\{g(k_r)\} \sim \tfrac{1}{2}J_0\{g(k_r)\} + \tfrac{1}{2}jY_0\{g(k_r)\}, \qquad r > 0, \tag{66}$$

so that

$$J_0\{g(k_r)\} \sim jY_0\{g(k_r)\}, \qquad r > 0. \tag{67}$$

Substituting this expression into Eq. (65), we find that

$$\mathcal{H}_u\{p_u(r)\} \sim \tfrac{1}{4}g(k_r) + \tfrac{1}{4}\mathbf{H}_0\{Y_0\{g(k_r)\}\} + \tfrac{1}{4}j[-jJ_0\{J_0\{g(k_r)\}\} - j\mathbf{H}_0\{Y_0\{g(k_r)\}\}], \tag{68}$$

which is valid for all k_r. Substituting the orthogonality relationships[12,13]

$$J_0\{J_0\{g(k_r)\}\} = g(k_r) \tag{69}$$

and

$$\mathbf{H}_0\{Y_0\{g(k_r)\}\} = \mathrm{sgn}(k_r)g(k_r) \tag{70}$$

into relation (68) justifies the statement that

$$\mathcal{H}_u\{p_u(r)\} \sim g(k_r)u(k_r). \tag{71}$$

Statement 3.4 summarizes the real-part–imaginary-part sufficiency condition that occurs in the k_r domain. The fact that $g(k_r)$ has an approximate real-part–imaginary-part sufficiency condition is not completely unexpected, since, as previously discussed, there exists an approximate causality condition in the alternate r domain. To justify statement 3.4, statement 3.3 is used to obtain the expression

$$\mathcal{H}_u\{p_u(r)\} \sim g(k_r)u(k_r). \tag{72}$$

Writing the operator $\mathcal{H}_u$ in terms of the operators J_0 and Y_0 yields

$$J_0\{\mathrm{Re}[p_u(r)]\} + \mathbf{H}_0\{\mathrm{Im}[p_u(r)]\} + j(-\mathbf{H}_0\{\mathrm{Re}[p_u(r)]\} + J_0\{\mathrm{Im}[p_u(r)]\}) \sim \mathrm{Re}[g(k_r)] + j\,\mathrm{Im}[g(k_r)], \qquad k_r > 0. \tag{73}$$

It is noted that the real and imaginary components on the left-hand side of this expression form a Hilbert transform pair, so that the real and imaginary components on the right-hand side are also related approximately by the Hilbert transform.

Although the specific statements made involve the relationships between the Hilbert–Hankel transform and the complex Hankel transform, it is also possible to derive a number of interesting relationships among the J_0, Y_0, and $\mathbf{H}_0$ transforms that compose these. Several of the relationships are derived in Appendix B.

The condition that $p(r) \sim p_u(r)$, $r > 0$ is restrictive in the context of the general class of circularly symmetric signals $p(r)$. For example, in Fig. 3, the positions of several poles in the k_r plane, corresponding to a rational function $g(k_r)$, are indicated. The poles labeled P_1', P_2', and P_3' are in symmetrically located positions with respect to poles P_1, P_2, and P_3 because $g(k_r)$ is even. The condition that $p(r) \sim p_u(r)$, $r > 0$ is equivalent to the statement that the bilateral inverse Hankel transform integration contour $C_1 + C_2$ can be approximately replaced by the contour C_1. Clearly, the approximation will be poor if a pole, such as P_3, is located in quadrant II of the k_r plane. Essentially, the effects of this pole, quite important in determining the character of the corresponding signal $p(r)$ for $r > 0$, are only negligibly included by integrating along the positive real axis only. Note that the pole at P_3' determines the behavior of the signal $p(r)$ primarily for values of $r < 0$. Thus, if $p(r)$ for $r > 0$ is exactly synthesized as

$$p(r) = \tfrac{1}{2}\int_{C_1+C_2} g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r, \tag{74}$$

so that

$$p(r) = \tfrac{1}{2}\int_{C_1} g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r + \tfrac{1}{2}\int_{C_2} g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r, \tag{75}$$

the pole at position P_3 contributes primarily to the second of these two integrals for values of $r > 0$. The approximation

$$p(r) \sim \tfrac{1}{2}\int_{C_1} g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r = \mathcal{H}_u^{-1}\{g(k_r)\} \tag{76}$$

is not accurate for $r > 0$ because of the position of pole P_3 in the k_r plane.

Although the condition that $p(r) \sim p_u(r)$, $r > 0$ is a restrictive condition for the general class of two-dimensional circularly symmetric functions $p(r)$, the condition is much less restrictive in the context of wave propagation. Essentially, the condition $p(r) \sim p_u(r)$, $r > 0$, when written in the form

$$p(r) \sim \tfrac{1}{2}\int_0^\infty g(k_r)H_0^{(1)}(k_r r)k_r \mathrm{d}k_r, \qquad r > 0, \tag{77}$$

can be interpreted as the statement that $p(r)$ is accurately approximated by a superposition of positive, or outgoing, wave-number components only. This is consistent with the fact that in wave propagation problems physically meaningful poles do not arise in quadrant II or IV of the k_r plane. The term outgoing has been associated with the function $H_0^{(1)}(k_r r)$, as can be justified by asymptotically expanding $H_0^{(1)}(k_r r)$, for $r \gg 0$. For example, a propagating wave field that is due to a harmonic point source can be written for $r \gg 0$ as

$$p(r, t) \sim \frac{1}{(2\pi)^{1/2}}\int_0^\infty \frac{g(k_r)\exp(-j\pi/4)}{(k_r r)^{1/2}}\exp[j(k_r r - \omega t)]k_r \mathrm{d}k_r. \tag{78}$$

The field is seen to be composed of the superposition of outgoing cylindrical waves of the form $\exp[j(k_r r - \omega t)]/(k_r r)^{1/2}$. It is also possible to write the error in the Hilbert–Hankel transform approximation as

$$\epsilon(r) \equiv p(r) - p_u(r) = \tfrac{1}{2} \int_0^\infty g(k_r) H_0^{(2)}(k_r r) k_r \mathrm{d}k_r, \qquad r > 0. \tag{79}$$

The error can be interpreted as a synthesis over the incoming wave-number components of $p(r)$. Here, the term incoming has been associated with the function $H_0^{(2)}(k_r r)$. This association can be justified by asymptotically expanding $H_0^{(2)}(k_r r)$, for $r \gg 0$. The error, with temporal variation included, can thus be written as

$$\epsilon(r, t) \sim \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{g(k_r)\exp(j\pi/4)}{(k_r r)^{1/2}} \exp[-j(k_r r + \omega t)] k_r \mathrm{d}k_r. \tag{80}$$

The error is seen to be composed of the superposition of incoming cylindrical waves of the form $\exp[-j(k_r r + \omega t)]/(k_r r)^{1/2}$.

The unilateral synthesis implied by the condition $p(r) \sim p_u(r)$, $r > 0$ is widely used in the area of underwater acoustics. For example, the unilateral synthesis implied by the Hilbert–Hankel transform is an important component in a number of synthetic data generation methods for acoustic fields, such as the fast-field program (FFP).[14,15] The implication is that the two-dimensional theory of approximately analytic signals, based on the condition $p(r) \sim p_u(r)$, $r > 0$, is applicable to the wide class of outwardly propagating wave fields.

To summarize briefly, in this section the property of approximate analyticity was extended to two-dimensional circularly symmetric signals. To do this, we developed a bilateral version of the inverse Hankel transform and its unilateral counterpart, referred to as the Hilbert–Hankel transform. Under the condition that the two-dimensional circularly symmetric signal is approximated by the Hilbert–Hankel transform for $r > 0$, it was shown that the real and imaginary parts of such a signal are approximately related. The Hilbert–Hankel transform was also related to another unilateral transform, referred to as the complex Hankel transform. A number of consequences based on the validity of the Hilbert–Hankel transform were developed. The theory is of particular importance because of its application to outgoing wave fields.

4. APPLICATION OF THE HILBERT-HANKEL TRANSFORM TO THE RECONSTRUCTION OF UNDERWATER ACOUSTIC FIELDS

In the previous section, the Hilbert–Hankel transform was defined and a number of its properties were developed. It was shown that if the causal portion of a circularly symmetric signal, described by the bilateral inverse Hankel transform, can be approximated by the Hilbert–Hankel transform, there are some important consequences that include an approximate real-part–imaginary-part sufficiency condition. As one application of the Hilbert–Hankel transform, we develop in this section an efficient reconstruction algorithm for obtaining a complex-valued underwater acoustic field from its real or imaginary part. Several examples of the reconstruction algorithm are provided in this section, and additional examples can be found in Ref. 16.

In underwater acoustics, the spatial part of the acoustic pressure field due to a point time-harmonic source located at $\mathbf{r}_0$ satisfies the Helmholtz equation

$$[\nabla^2 + k^2(\mathbf{r})]p(\mathbf{r}, \mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0), \tag{81}$$

where $k(\mathbf{r}) = \omega/c(\mathbf{r})$, ω is the frequency of the source and $c(\mathbf{r})$ is the speed of sound. If horizontal stratification is assumed, with the source located at $(0, z_0)$, the solution to Eq. (81) can be written as

$$p(r, z; z_0) = \int_0^\infty g(k_r, z; z_0) J_0(k_r r) k_r \mathrm{d}k_r. \tag{82}$$

The function $g(k_r, z; z_0)$, referred to as the depth-dependent Green's function, satisfies the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + k^2(z) - k_r^2\right) g(k_r, z; z_0) = -2\delta(z - z_0). \tag{83}$$

In a number of applications, measurements of the complex-valued field $p(r, z; z_0)$ are required. For example, in the inverse problem of determining the ocean bottom reflection coefficient, the Hankel transform of the complex-valued field, for fixed values of z and z_0, is determined to yield the depth-dependent Green's function, and the reflection coefficient is then extracted.[17,18] We will show that both the magnitude and the phase of the acoustic field $p(r)$ can be approximated from measurements of a single quadrature component only.

In Section 3, a real-part–imaginary-part sufficiency condition was obtained by requiring that $p(r) \sim p_u(r)$ for $r > 0$, where

$$p(r) = \int_0^\infty g(k_r) J_0(k_r r) k_r \mathrm{d}k_r \tag{84}$$

and

$$p_u(r) = \tfrac{1}{2} \int_0^\infty g(k_r) H_0^{(1)}(k_r r) k_r \mathrm{d}k_r. \tag{85}$$

It was also shown that under the condition that $p(r) \sim p_u(r)$ for $r > 0$, the function $p_u(r)$ is approximately a causal function of r, so that

$$p(r)u(r) \sim \tfrac{1}{2} \int_0^\infty g(k_r) H_0^{(1)}(k_r r) k_r \mathrm{d}k_r. \tag{86}$$

We will find it convenient to substitute the asymptotic expansion for the Hankel function,

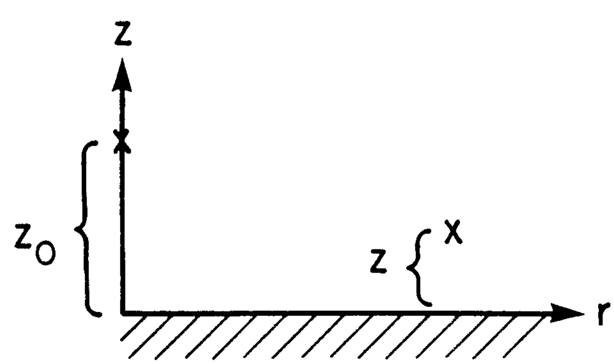

Fig. 4. Geoacoustic model for the Lloyd mirror field corresponding to the sum of a direct field plus a field that has interacted with a pressure-release bottom. Parameters: $R_B = -1$, $z_0 = 100$ m, $z = 50$ m, $f = 220$ Hz, $c = 1500$ m/sec, $k_0 = 0.9215$ rad/m.

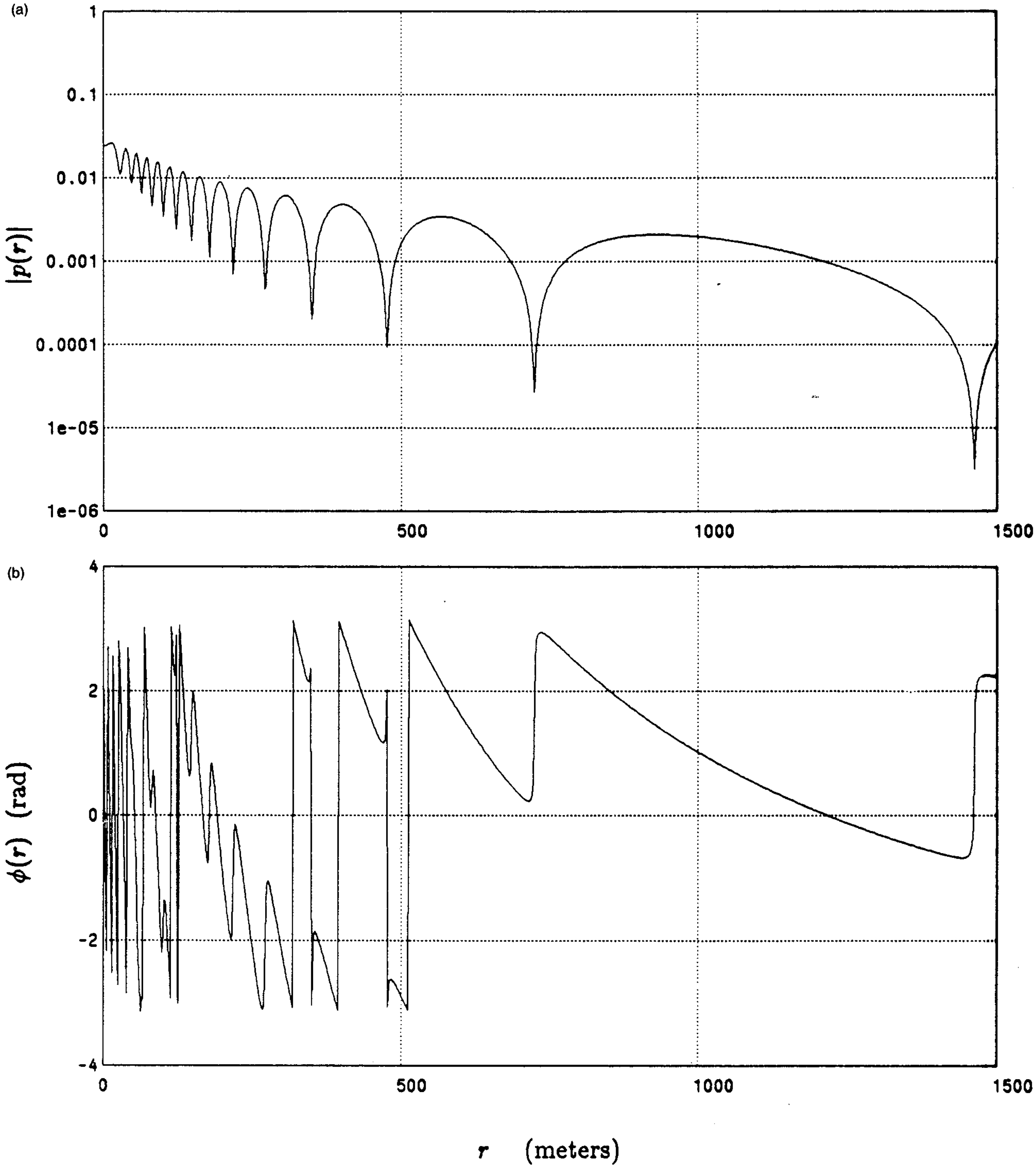

Fig. 5. (a) Magnitude and (b) residual phase of the Lloyd mirror field.

$$H_0^{(1)}(k_r r) \sim \left(\frac{2}{\pi k_r r}\right)^{1/2} \exp[j(k_r r - \pi/4)], \qquad (87)$$

into relation (86) to yield

$$p(r)u(r) \sim \frac{1}{(2\pi)^{1/2}} \int_0^\infty g(k_r) \frac{\exp[j(k_r r - \pi/4)]}{(k_r r)^{1/2}} k_r \mathrm{d}k_r. \qquad (88)$$

Multiplying both sides of this equation by $r^{1/2}$ and defining $\tilde{g}(k_r)$ as

$$\tilde{g}(k_r) = (2\pi k_r)^{1/2} g(k_r) \exp(-j\pi/4), \qquad (89)$$

we find that

$$p(r)r^{1/2}u(r) \sim \frac{1}{2\pi} \int_0^\infty \tilde{g}(k_r) \exp(jk_r r) \mathrm{d}k_r, \qquad r \gg 0. \quad (90)$$

This expression is the basis of the FFP commonly used in underwater acoustics for synthetic data generation.[14,15,19] Relation (90) also implies that $p(r)r^{1/2}u(r)$ is approximately analytic, since it is approximately synthesized in terms of the unilateral inverse Fourier transform. Thus, by incorporating the additional asymptotic expansion for $H_0^{(1)}(k_r r)$, the signals $\mathrm{Re}[p(r)r^{1/2}u(r)]$ and $\mathrm{Im}[p(r)r^{1/2}u(r)]$ are seen to be approximately related by the Hilbert transform for $r \gg 0$.

When sampled versions of these signals are involved, there exist several methods for determining the Hilbert transform.[20,21] In the method that we chose, a sampled version of $\mathrm{Re}[p(r)u(r)r^{1/2}] + j\,\mathrm{Im}[p(r)u(r)r^{1/2}]$ is obtained by computing the fast Fourier transform (FFT) of $\mathrm{Re}[p(r)u(r)r^{1/2}]$ (or $j\,\mathrm{Im}[p(r)u(r)r^{1/2}]$), multiplying by $2u(k_r)$, and computing the inverse FFT. A discrete Hilbert transform based on an optimal finite impulse response filter design may also have application to this problem.[22,23]

We will now present three examples of this reconstruction method. The first example consists of a simple acoustic field that was synthetically generated by using a closed-form expression. The second example consists of a realistic deep-water acoustic that was synthetically generated by computing the Hankel transform of the associated Green's function. The third example consists of an acoustic field collected in an actual ocean experiment.

As the first example, consider the Lloyd mirror acoustic

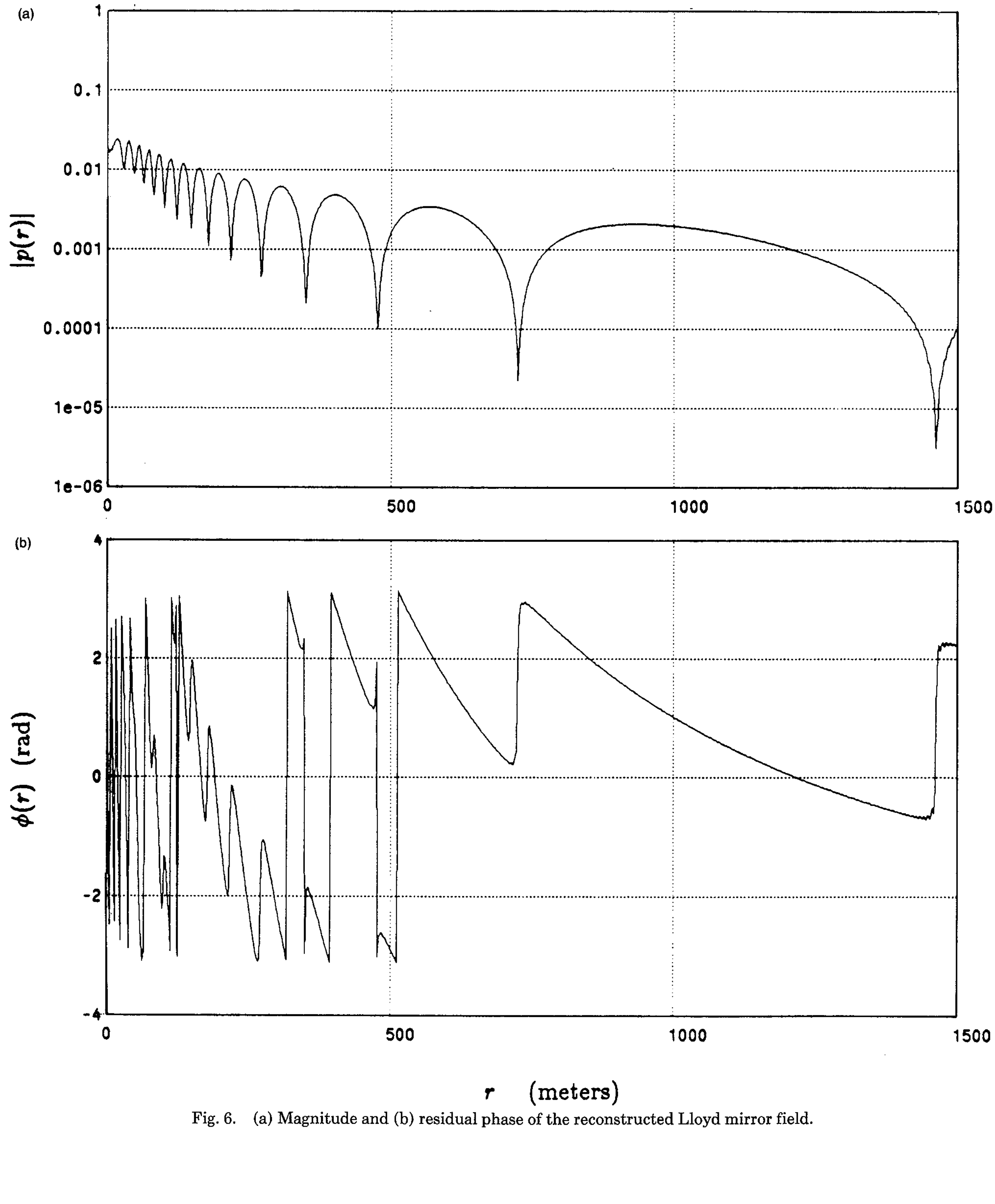

Fig. 6. (a) Magnitude and (b) residual phase of the reconstructed Lloyd mirror field.

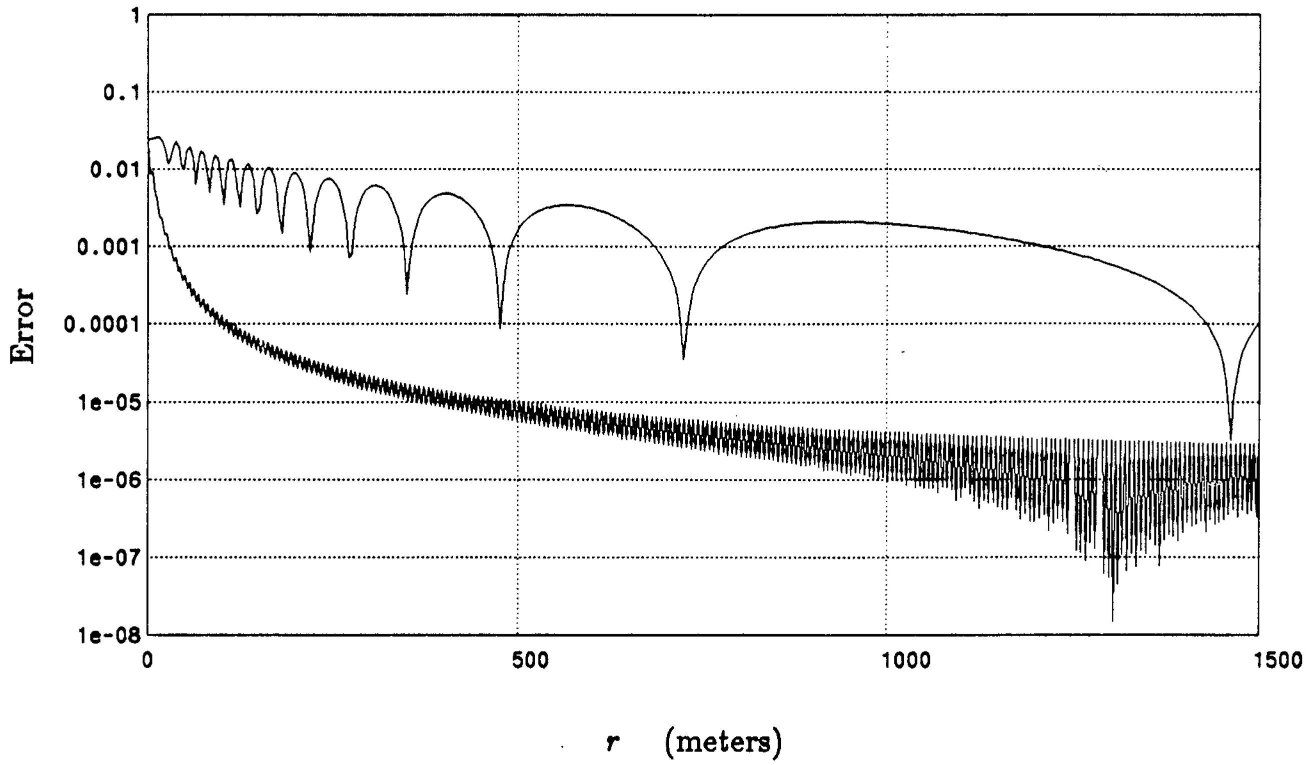

Fig. 7. Error in the reconstruction of the real component (bottom curve) and true field magnitude (top curve) of the Lloyd mirror field.

field,[24] which can be written as the sum of a direct field plus a reflected field that has interacted with a pressure release boundary. The total field can be written in closed form as

$$p(r) = \exp(jk_0R_0)/R_0 - \exp(jk_0R_1)/R_1, \tag{91}$$

where $R_0 = (r^2 + (z - z_0)^2)^{1/2}$, $R_1 = (r^2 + (z + z_0))^{1/2}$, and $k_0 = \omega/c_0$ is the water wave number. The geoacoustic model for this example is summarized in Fig. 4.

In Fig. 5(a) is shown the magnitude of the Lloyd mirror field. The peaks and nulls in the magnitude are characteristic of the interference between the direct field and the reflected field. Rather than displaying the rapidly varying phase of the field, we have chosen to display in Fig. 5(b) the related quantity $\phi(r)$, referred to as the residual phase,[16,25] which is defined as

$$\phi(r) \equiv P\{\arg\{p(r)\} - k_0r\}, \tag{92}$$

where $P\{\ \}$ denotes principal value. The residual phase is more slowly varying than the total phase because the linear contribution to the total phase is removed.

To perform the reconstruction, the acoustic field was sampled every 1.7 m, corresponding to a rate of approximately four samples per water wavelength. The real part of this field was set to zero, and 1024 samples of the imaginary part were retained. The real part was then reconstructed, and the magnitude and the residual phase of the reconstructed field are shown in Fig. 6. The reconstruction is excellent, as can be seen by comparing Figs. 5 and 6. As further evidence of the high quality of the reconstruction, the magnitude of the difference between the original real component and the reconstructed real component is plotted in Fig. 7 along with the original field magnitude. From this figure it can be seen that the error in the reconstruction occurs primarily in the near field.

The degradation in the reconstructed near field can be attributed to two effects. First, the assumption that the acoustic field can be synthesized by using the Hilbert–Hankel transform, i.e., using positive wave numbers only, is not strictly valid at very short ranges. Second, the asymptotic expressions, obtained from the asymptotic Hilbert–Hankel transform, that are used in the reconstruction method are not valid at short ranges. In the following two examples, we will see that the reconstruction method also yields some degradation in the near field. In many coherent processing applications this error may not be significant, but we point out that it is a limitation of the theory and method for reconstruction that we have proposed.

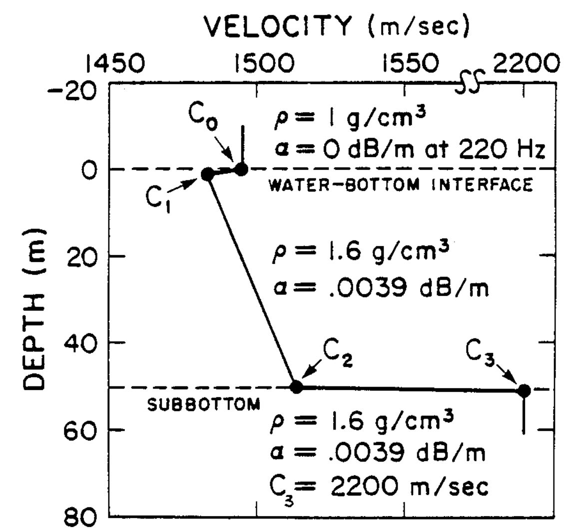

Fig. 8. Deep-water geoacoustic model. Additional parameters: $z_0 = 124.9$ m, $z = 1.2$ m, $f = 220$ Hz.

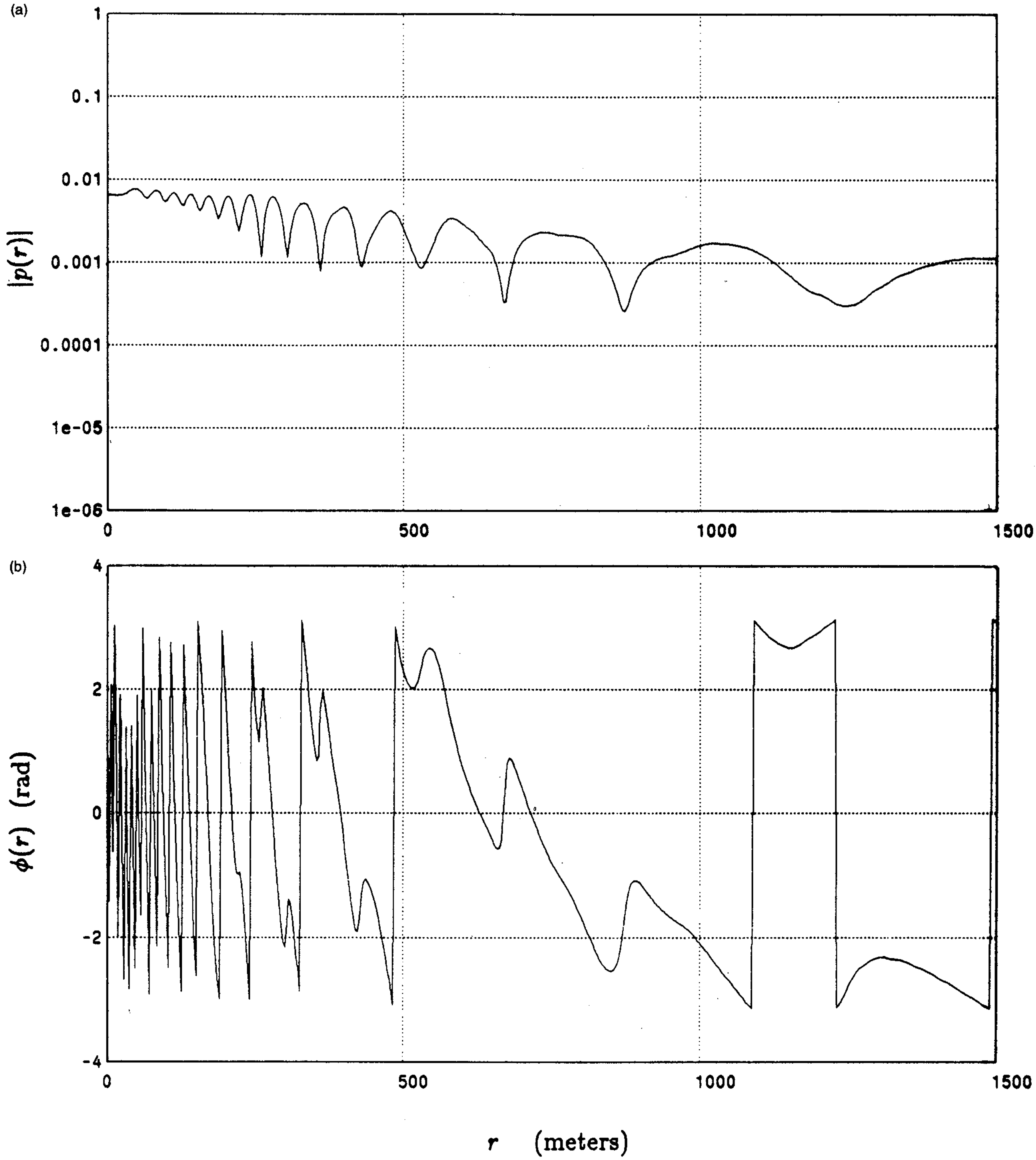

Fig. 9. (a) Magnitude and (b) residual phase of realistic deep-water field.

As the second example, we consider a synthetic deep-water acoustic field, which consists of a direct field plus a reflected field that has interacted with a horizontally stratified ocean bottom with a realistic sound velocity profile. The geoacoustic model for this example is summarized in Fig. 8. The synthetic field was generated by first computing the depth-dependent Green's function[16] and then computing its numerical Hankel transform. The resulting reflected field was then added to the closed-form expression for the direct field to obtain the total field. The magnitude and the residual phase of this field are shown in Fig. 9.

To perform the reconstruction, the real part of the field was set to zero, and 1024 samples of the imaginary part, which was sampled every 3.14 m, were retained. The real part was reconstructed, and the corresponding magnitude and residual phase are shown in Fig. 10. Comparing Figs. 9 and 10, we see that the reconstructed field is very similar to the original field. As further evidence of this, the error in the reconstructed real component is plotted in Fig. 11, along with the magnitude of the original field. The error signal is substantially smaller than the original field magnitude for ranges greater than about 25 m. The error in the reconstruction in the near field can again be attributed to the fact that the Hilbert–Hankel transform does not correctly synthesize the field at short ranges and also to the fact that the asymptotic expansion of the Hankel function is not valid at short ranges.

As the final example, we will consider the reconstruction of an acoustic field collected in a shallow-water ocean experiment, conducted in Nantucket Sound in May 1984 by the Woods Hole Oceanographic Institution.[26] The experimental configuration is shown in Fig. 12. The procedure for acquiring the acoustic data consisted of towing a harmonic source at a fixed depth away from the moored hydrophone receivers over an aperture of approximately 1320 m. The hydrophone receivers demodulated the harmonic pressure signal and digitally recorded both the real and the imaginary components of the spatial part of the field.

The magnitude and the unwrapped residual phase of the acoustic field recorded at the upper hydrophone are shown in Fig. 13. The interference pattern in the magnitude is due to the constructive and destructive interference of several trapped and partially trapped modes in the shallow-water waveguide.[26] The original acoustic field was sampled non-

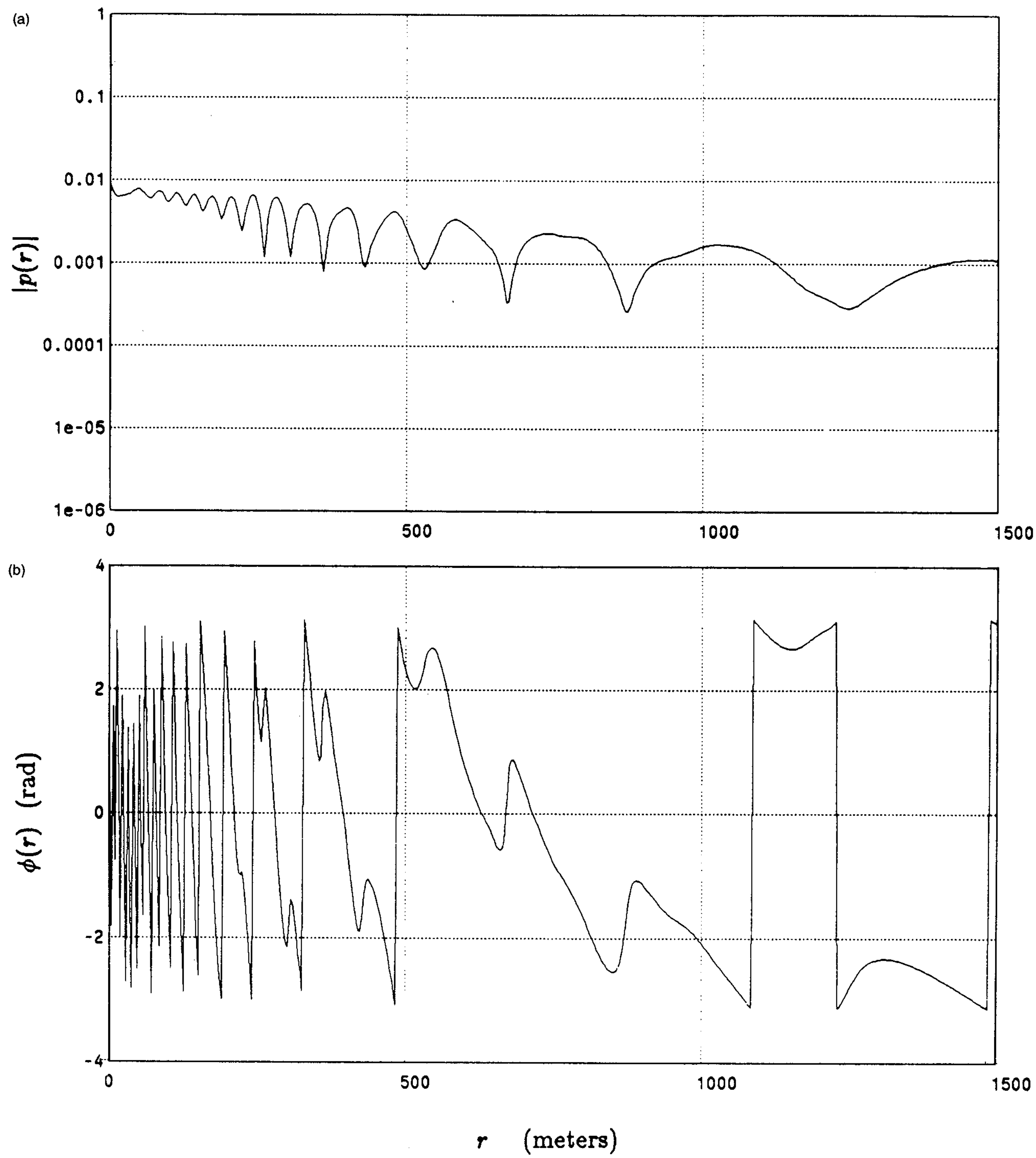

Fig. 10. (a) Magnitude and (b) residual phase of reconstructed realistic deep-water field.

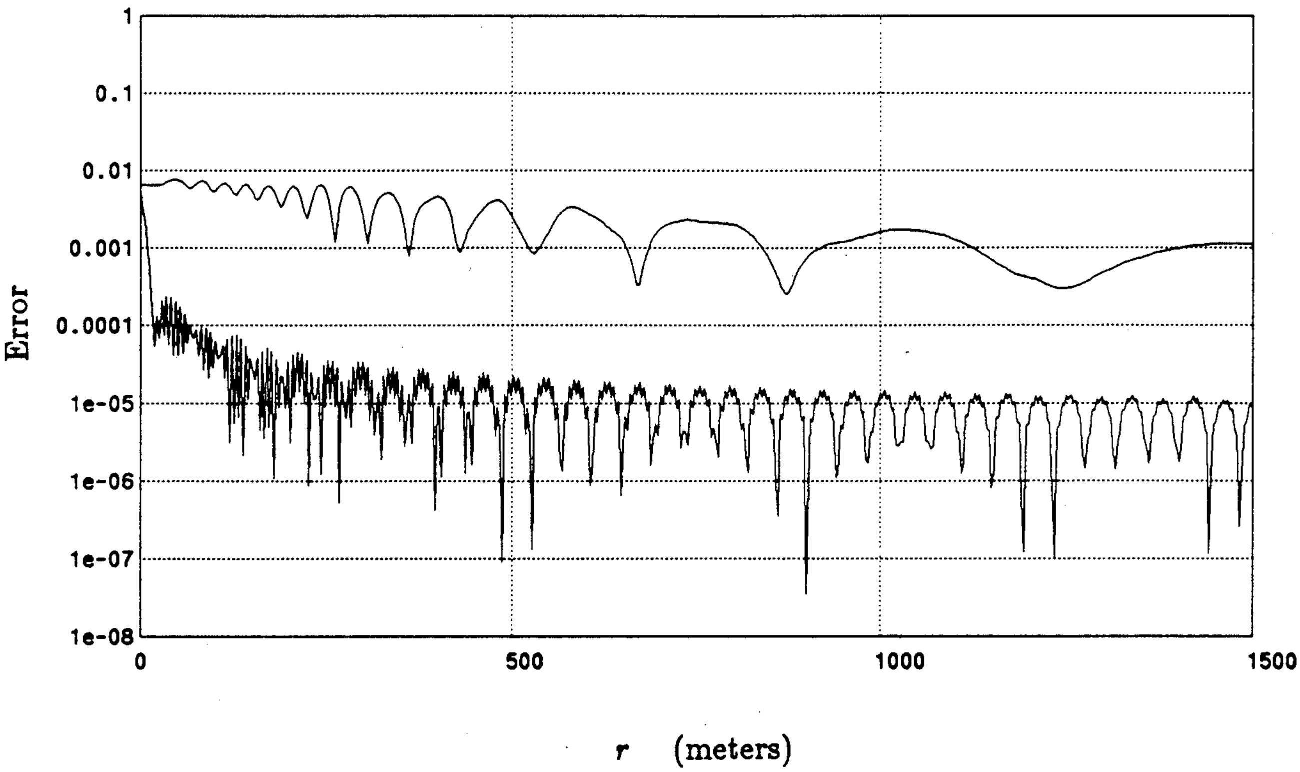

Fig. 11. Error in the reconstruction of the real component (bottom curve) and true field magnitude (top curve) for the realistic deep-water field.

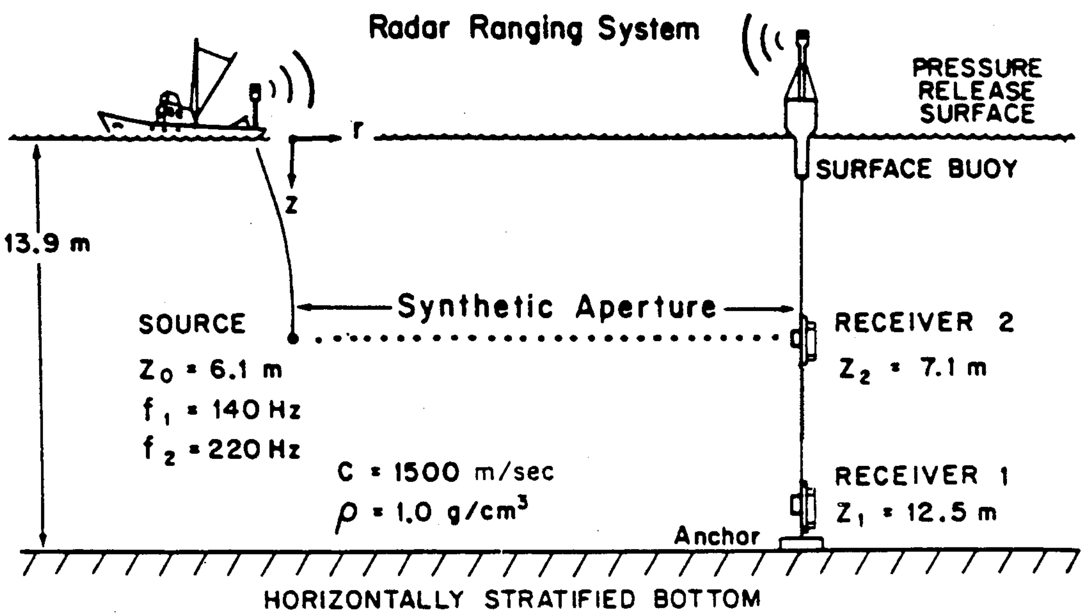

Fig. 12. Configuration of the ocean experiment.

uniformly in range because of the difficulty in maintaining a fixed spatial sampling interval in the ocean experiment. First the field was interpolated to a uniformly sampled grid before the reconstruction experiment.

To demonstrate the reconstruction algorithm, the real part of this experimental field was set to zero, and 1024 samples of the imaginary part were retained. The real part was then reconstructed from the imaginary part, and the corresponding magnitude and the unwrapped residual phase of the reconstructed field are shown in Fig. 14. The similarity of the reconstructed field with the original and resampled fields is evident. In particular, the behavior of the unwrapped residual phase is preserved, as are the peaks and nulls in the magnitude of the field. If the curves are examined in further detail, it can be seen that some of the finer details differ. These slight differences occur not only at near ranges, as was the case for the previous two examples, but at far ranges as well. These differences are also evident from examination of the error signal for this example, shown in Fig. 15.

There are several factors that may explain the reconstruction error, which is present not only at small values of range but at large values of range as well. Theoretically, the reconstruction algorithm is based on the real-part–imaginary-part sufficiency condition that applies to outgoing fields. In an ocean environment, the effects of velocity changes within the water column, as well as violations of both the horizontal stratification and radial symmetry, may yield acoustic fields that are not completely outgoing. Additionally, in shallow water, the effects of a nonperfect surface may yield scattered acoustic components that are not outgoing.

Nevertheless, the reconstructed experimental field is quite similar to the original field. In fact, the agreement between the curves suggests that this method might also be used as a measure of the quality of the experimental data collected. In this approach, both components are recorded and resampled to a uniform grid. One component is then set to zero and reconstructed from the alternate component. A

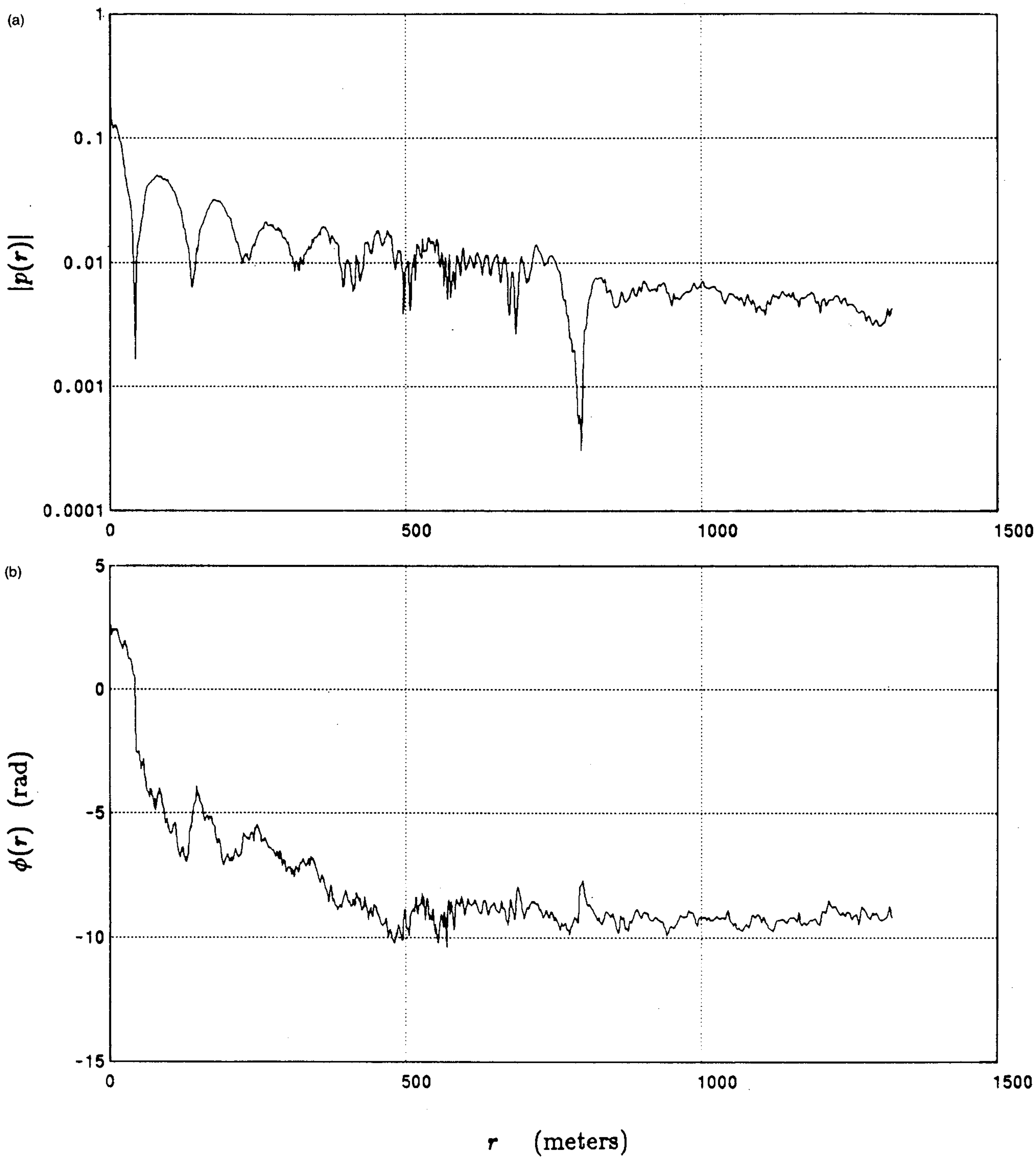

Fig. 13. (a) Magnitude and (b) unwrapped residual phase of the interpolated 140-Hz experimental shallow-water field.

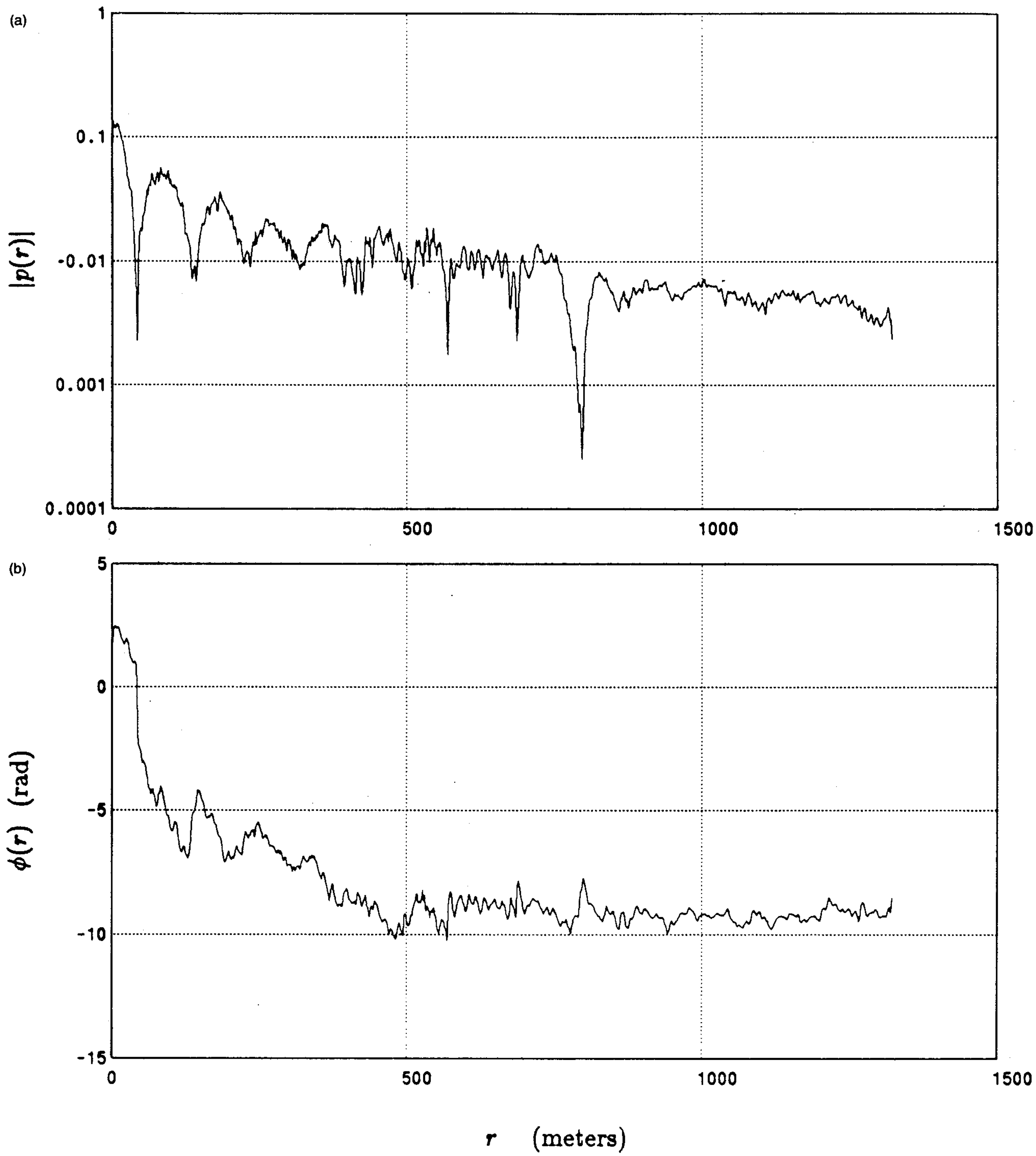

Fig. 14. (a) Magnitude and (b) unwrapped residual phase of the reconstructed 140-Hz experimental shallow-water field.

similar procedure can be performed with the alternate component. Presumably, if the reconstructed and original fields differ substantially, there is an implication that effects such as errors in range registration, surface scattering, and medium inhomogeneity cannot be treated as negligible. Such a procedure may also provide important information about the scattering properties of the medium as a function of position. In numerical experiments, we have seen a correlation in the quality of the reconstruction with the accuracy of the ranging method established independently. Thus we see that, in addition to providing a means for eliminating hardware in the data acquisition system, the reconstruction method can provide an important consistency check on the quality of the sampled field.

5. SUMMARY

In this paper we developed the theory for one- and two-dimensional unilateral transforms. In one dimension, it was shown that if the causal portion of an even signal can be approximately synthesized by the unilateral inverse Fourier transform, there are important consequences. Specifically, such a signal possesses an approximate real-part–imaginary-part sufficiency condition, as does its Fourier transform. Additionally, the approximation was shown to imply an inverse relationship between the unilateral inverse Fourier transform and the unilateral Fourier transform as well as other relationships between the cosine and sine transforms of the real and imaginary components of the signal and its Fourier transform.

In extending the theory to two dimensions, we developed a bilateral version of the Hankel transform and a unilateral version, referred to as the Hilbert–Hankel transform. It was shown that if the causal portion of a circularly symmetric signal can be approximately synthesized by the Hilbert–Hankel transform, there are important consequences. Specifically, such a signal possesses an approximate real-part–imaginary-part sufficiency condition, as does its Hankel transform. Additionally, the approximation implies an inverse relationship between the Hilbert–Hankel transform and the complex Hankel transform as well as other relationships among the J_0, Y_0, and $\mathbf{H}_0$ transforms of the real and imaginary components of the signal and its Hankel transform. As was pointed out, the unilateral synthesis in two dimensions is particularly applicable to circularly symmetric fields that are outwardly propagating. The two-dimensional result led to the development of a reconstruction algorithm for obtaining the real (or imaginary) component from the imaginary (or real) component of a circularly symmetric field. This algorithm was applied to three examples of underwater acoustic pressure fields.

Although our interest has been primarily in the application of the two-dimensional theory to the underwater acoustics problem, the general nature of the theory suggests that it may be applicable to other problems as well. For example, one application of the one-dimensional theory might be to the propagation of a wide-band acoustic pulse in a seismic borehole in which the upgoing field is approximately synthesized by the unilateral inverse Fourier transform.[27]

Also, the development of the general multidimensional theory incorporating different symmetry conditions is suggested. This includes, for example, two-dimensional even signals that can be approximated by the unilateral inverse two-dimensional Fourier transform. Although we have developed the theory for an approximate real-part–imaginary-part sufficiency condition, it is also possible to develop the analogous theory for an approximate magnitude-phase sufficiency condition by applying the complex logarithm to the Fourier transform.[21] Essentially, in one dimension there exists duality between the even signal and the even cepstrum and between the approximate real-part–imaginary-part sufficiency condition and the approximate magnitude-phase sufficiency condition. The duality suggests that there exists an approximate Hilbert transform relationship between the magnitude and the phase for a special class of mixed-phase signals. The relationship between the unilateral transform and cepstrum may have some important consequences in applications that include reconstruction of a signal from its magnitude only, phase unwrapping, and general homomorphic signal processing.

APPENDIX A

To develop additional relationships between the cosine and sine transforms, we introduce the following notation:

$$C\{F(\omega)\} \equiv \frac{1}{\pi}\int_0^\infty F(\omega)\cos \omega t d\omega, \tag{A1}$$

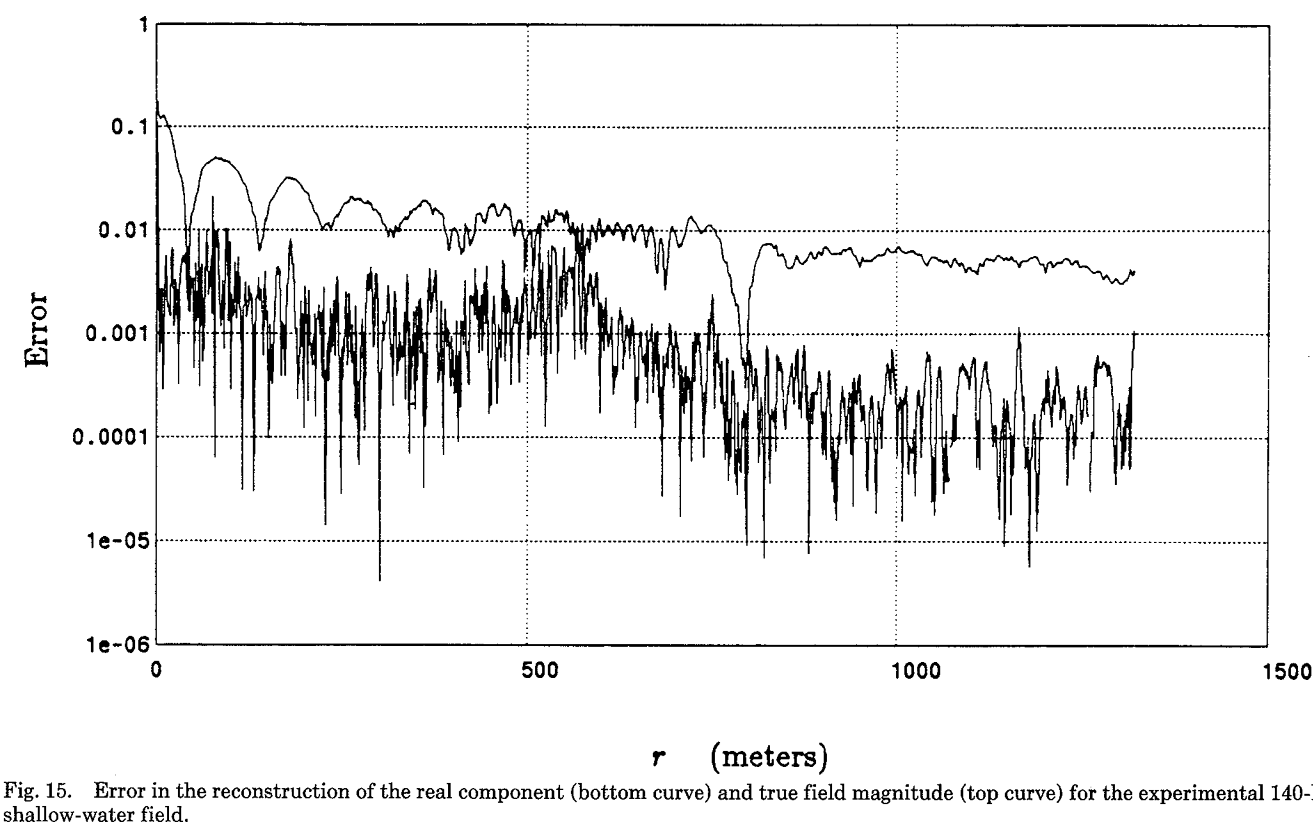

Fig. 15. Error in the reconstruction of the real component (bottom curve) and true field magnitude (top curve) for the experimental 140-Hz shallow-water field.

$$S\{F(\omega)\} \equiv \frac{1}{\pi}\int_0^\infty F(\omega)\sin \omega t d\omega. \quad (A2)$$

Note that since the signal $f(t)$ is even, and thus its Fourier transform $F(\omega)$ is also even, Eqs. (13) and (15) in the text can be written in terms of cosine transforms as

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = C\{F(\omega)\}, \quad (A3)$$

$$F(\omega) = \mathcal{F}\{f(t)\} = 2\pi C\{f(t)\}. \quad (A4)$$

Additionally, the unilateral inverse Fourier transform and the unilateral Fourier transform, in Eqs. (14) and (16) of the text, can be written in terms of cosine and sine transforms as

$$f_u(t) = \mathcal{F}_u^{-1}\{F(\omega)\} = \tfrac{1}{2}C\{F(\omega)\} + j\tfrac{1}{2}S\{F(\omega)\} \quad (A5)$$

and

$$F_u(\omega) = \mathcal{F}_u\{f(t)\} = \pi C\{f(t)\} - j\pi S\{f(t)\}. \quad (A6)$$

Two statements involving the relationships between the various cosine and sine transforms are now made.

Statement A1 *If $f(t) \sim f_u(t)$ for $t > 0$, then the cosine and sine transforms of the real and imaginary components of $F(\omega)$ are related, for $t > 0$, by*

$$C\{\mathrm{Re}[F(\omega)]\} \sim -S\{\mathrm{Im}[F(\omega)]\}, \quad (A7)$$

$$C\{\mathrm{Im}[F(\omega)]\} \sim S\{\mathrm{Re}[F(\omega)]\}. \quad (A8)$$

Additionally, the cosine and sine transforms of the real and imaginary components of $f(t)$ are related, for $\omega > 0$, by

$$C\{\mathrm{Re}[f(t)]\} \sim S\{\mathrm{Im}[f(t)]\}, \quad (A9)$$

$$C\{\mathrm{Im}[f(t)]\} \sim -S\{\mathrm{Re}[f(t)]\}. \quad (A10)$$

To derive the first pair of equations, the fact that $f(t) \sim f_u(t)$, $t > 0$ implies that

$$\mathcal{F}^{-1}\{F(\omega)\} \sim \mathcal{F}_u^{-1}\{F(\omega)\}, \qquad t > 0 \quad (A11)$$

is used. Substituting Eq. (A3) into the left-hand portion of the expression and Eq. (A5) into the right-hand portion of the expression and equating real and imaginary parts on both sides yields the first pair of equations. To derive the second pair of equations, we utilize statement 2.3, which relates the unilateral inverse Fourier transform and the unilateral Fourier transform, to derive that

$$\mathcal{F}\{f(t)\} \sim \mathcal{F}_u\{f(t)\}, \qquad \omega > 0. \quad (A12)$$

Substituting Eq. (A4) into the left-hand portion of the expression and Eq. (6) into the right-hand portion of the expression and equating real and imaginary parts on both sides yields the second pair of equations in the statement.

Another interesting consequence of the validity of the unilateral synthesis of $f(t)$ for $t > 0$ is presented in the following statement.

Statement A2 *If $f(t) \sim f_u(t)$ for $t > 0$, then $f(t)$ can be approximately synthesized, for $t > 0$, in terms of either the real or the imaginary components of $F(\omega)$, as*

$$f(t) \sim 2\mathcal{F}_u^{-1}\{\mathrm{Re}[F(\omega)]\}, \quad (A13)$$

$$f(t) \sim 2j\mathcal{F}_u^{-1}\{\mathrm{Im}[F(\omega)]\}. \quad (A14)$$

Additionally, $F(\omega)$ can be approximately analyzed, for $\omega > 0$, in terms of either the real or the imaginary components of $f(t)$, as

$$F(\omega) \sim 2\mathcal{F}_u\{\mathrm{Re}[f(t)]\}, \quad (A15)$$

$$F(\omega) \sim 2j\mathcal{F}_u\{\mathrm{Im}[f(t)]\}. \quad (A16)$$

These relationships may be of importance if only one component of the signal (or Fourier transform) is available and it is desirable to determine the Fourier transform (or signal). To obtain expressions (A13) and (A14), the fact that $f(t) \sim f_u(t)$, $t > 0$ implies that

$$f(t) \sim \mathcal{F}_u^{-1}\{F(\omega)\}, \qquad t > 0 \quad (A17)$$

is used. The right-hand side of this relation is expressed in terms of cosine and sine transforms, as in Eq. (A5). The relationships stated in the first part of statement A1 are then substituted to derive expressions (A13) and (A14). To obtain the second pair of equations, we utilize statement 2.3, which relates the unilateral inverse Fourier transform and the unilateral Fourier transform, to derive that

$$F(\omega) \sim \mathcal{F}_u\{f(t)\}, \qquad \omega > 0. \quad (A18)$$

The right-hand side of this relation is then expressed in terms of cosine and sine transforms, using Eq. (A6). The relationships stated in the second part of statement A1 are then substituted to derive expressions (A15) and (A16).

APPENDIX B

To develop additional relationships among the J_0, Y_0, and $\mathbf{H}_0$ transforms, we write the Hilbert–Hankel transform and the complex Hankel transform as

$$p_u(r) = \mathcal{H}_u^{-1}\{g(k_r)\} = \tfrac{1}{2}J_0\{g(k_r)\} + j\tfrac{1}{2}Y_0\{g(k_r)\} \quad (B1)$$

and

$$g_u(k_r) = \mathcal{H}_u\{p(r)\} = \tfrac{1}{2}J_0\{p(r)\} - j\tfrac{1}{2}\mathbf{H}_0\{p(r)\}. \quad (B2)$$

Two statements involving the relationships among the J_0, Y_0, and $\mathbf{H}_0$ transforms are now made.

Statement B1 *If $p(r) \sim p_u(r)$ for $r > 0$, then the J_0 and Y_0 transforms of the real and imaginary components of $g(k_r)$ are related for $r > 0$ by*

$$J_0\{\mathrm{Re}[g(k_r)]\} \sim -Y_0\{\mathrm{Im}[g(k_r)]\}, \quad (B3)$$

$$J_0\{\mathrm{Im}[g(k_r)]\} \sim Y_0\{\mathrm{Re}[g(k_r)]\}. \quad (B4)$$

Additionally, the J_0 and $\mathbf{H}_0$ transforms of the real and imaginary components of $p(r)$ are related, for $k_r > 0$, by

$$J_0\{\mathrm{Re}[p(r)]\} \sim \mathbf{H}_0\{\mathrm{Im}[p(r)]\}, \quad (B5)$$

$$J_0\{\mathrm{Im}[p(r)]\} \sim -\mathbf{H}_0\{\mathrm{Re}[p(r)]\}. \quad (B6)$$

To justify the first part of this statement, the condition $p(r) \sim p_u(r)$, $r > 0$ is written as

$$J_0\{g(k_r)\} \sim \tfrac{1}{2}J_0\{g(k_r)\} + j\tfrac{1}{2}Y_0\{g(k_r)\}, \qquad r > 0, \quad (B7)$$

so that

$$J_0\{g(k_r)\} \sim jY_0\{g(k_r)\}, \qquad r > 0. \quad (B8)$$

If the real and imaginary parts on both sides of this expression are equated, the first pair of expressions in the statement are obtained. To derive the second pair of expressions, we use statement 3.3, which relates the Hilbert–Hankel transform and the complex Hankel transform, to derive that

$$J_0\{p(r)\} \sim \mathcal{H}_u\{p(r)\}, \qquad k_r > 0. \tag{B9}$$

Using Eq. (B2), this expression becomes

$$J_0\{g(k_r)\} \sim {}^1\!/_2 J_0\{p(r)\} - j{}^1\!/_2\mathbf{H}_0\{p(r)\}, \qquad k_r > 0, \tag{B10}$$

so that

$$J_0\{p(r)\} \sim -j\mathbf{H}_0\{p(r)\}, \qquad k_r > 0. \tag{B11}$$

Equating the real and imaginary parts on both sides of this expression yields the second pair of expressions.

An additional consequence of the validity of the unilateral synthesis of $p(r)$ for $r > 0$ is summarized in the following statement.

Statement B2 *If $p(r) \sim p_u(r)$ for $r > 0$, then $p(r)$ can be approximately synthesized, for $r > 0$, in terms of either the real or the imaginary components of $g(k_r)$, as*

$$p(r) \sim 2\mathcal{H}_u^{-1}\{\mathrm{Re}[g(k_r)]\}, \tag{B12}$$

$$p(r) \sim 2j\mathcal{H}_u^{-1}\{\mathrm{Im}[g(k_r)]\}. \tag{B13}$$

Additionally, $g(k_r)$ can be approximately analyzed, for $k_r > 0$, in terms of either the real or the imaginary components of $p(r)$, as

$$g(k_r) \sim 2\mathcal{H}_u\{\mathrm{Re}[p(r)]\}, \tag{B14}$$

$$g(k_r) \sim 2j\mathcal{H}_u\{\mathrm{Im}[p(r)]\}. \tag{B15}$$

To develop the first pair of expressions, we use the fact that

$$p(r) \sim {}^1\!/_2 J_0\{g(k_r)\} + j{}^1\!/_2 Y_0\{g(k_r)\}, \qquad r > 0. \tag{B16}$$

If expressions (B3) and (B4) are substituted into the right-hand side of this expression, the first pair of expressions is obtained. To derive the second pair, we use statement 3.3, which relates the Hilbert–Hankel transform and the complex Hankel transform, to derive that

$$g(k_r) \sim \mathcal{H}_u\{p(r)\}, \qquad k_r > 0. \tag{17}$$

If expressions (B5) and (B6) are substituted into the right-hand side of this expression, the second pair of expressions is obtained.

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