RECONSTRUCTION OF PROPAGATING COMPLEX WAVE FIELDS USING THE HILBERT-HANKEL TRANSFORM [1]

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ABSTRACT

In this paper, an approximate real-part sufficiency condition is developed for complex-valued one-dimensional even signals and two-dimensional circularly symmetric signals. The two-dimensional result is used in a reconstruction algorithm which is applied to synthetic and experimental underwater acoustic fields.

INTRODUCTION

A well-known property in Fourier transform theory is that causality in one domain implies real-part sufficiency in the alternate domain. This property is the basis for the fact that the real and imaginary components of a signal are related via the Hilbert transform, if the spectrum of the signal is causal. In wave propagation problems, it is often the circularly symmetric two-dimensional Fourier transform, or equivalently, the Hankel transform which is of central importance. Because of the symmetry in such problems, the condition of causality is not applicable. In one dimension, the counterpart of the circularly symmetric signal is the even signal. The one-dimensional Fourier transform of an even signal is also even, and thus the condition of causality is not applicable in this case as well.

In our work, we have shown that under some conditions it is possible to approximately relate the real and imaginary components of a one-dimensional even signal, or a two-dimensional circularly symmetric signal. The approximation is based on the validity of the unilateral inverse Fourier transform in one dimension, and a unilateral version of the Hankel transform in two dimensions. In this paper, we develop the approximate real-part sufficiency condition in one and two dimensions using these transforms. The two-dimensional result forms the basis for a reconstruction algorithm in which the real (or imaginary) component of an underwater acoustic field is obtained from the imaginary (or real) component.

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ONE-DIMENSIONAL THEORY

Consider an even signal $f(t)$ which has the Fourier transform $F(\omega)$, where

$$f(t) \equiv \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad (1)$$

The unilateral inverse Fourier transform is defined as

$$f_u(t) \equiv \mathcal{F}_u^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{0}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad (2)$$

There exists an approximate relationship between the real and imaginary components of $f(t)$ if the condition

$$f(t) \sim f_u(t) \;\; t > 0 \qquad (3)$$

is satisfied. To develop the relationship, the signals $f_u(t)$ and $f(t)$ are written in terms of cosine and sine transforms as

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty} F(\omega)\cos\omega t d\omega \qquad (4)$$

and

$$f_u(t) = \frac{1}{2\pi}\int_{0}^{\infty} F(\omega)\cos\omega t d\omega + \frac{j}{2\pi}\int_{0}^{\infty} F(\omega)\sin\omega t d\omega \qquad (5)$$

The condition in equation (3) therefore implies that

$$\frac{1}{2\pi}\int_{0}^{\infty} F(\omega)\cos\omega t d\omega \sim \frac{j}{2\pi}\int_{0}^{\infty} F(\omega)\sin\omega t d\omega \qquad (6)$$

By equating real and imaginary parts on both sides of this expression, it can be shown that for $t > 0$

$$Re[f(t)] \sim -\frac{2}{\pi}\int_0^\infty \sin\omega t d\omega \int_0^\infty Im[f(t)]\cos\omega t dt \quad (7)$$

$$Im[f(t)] \sim \frac{2}{\pi}\int_0^\infty \sin\omega t d\omega \int_0^\infty Re[f(t)]\cos\omega t dt \quad (8)$$

To the extent that the approximation in equation (3) is valid, it can also be shown [1] that there is an inverse relationship between the unilateral inverse Fourier transform and the unilateral Fourier transform.

The condition $f(t) \sim f_u(t), t > 0$ is quite restrictive and does not apply to an arbitrary even signal $f(t)$. It implies, for example, that there can be no poles located in Quadrant I or III of the Laplace transform s-plane. However, the corresponding theory in the circularly symmetric two-dimensional case is less restrictive. In particular, as will be shown in the following section, the condition applies to the general class of circularly symmetric signals which are related to outwardly propagating wave fields.

TWO-DIMENSIONAL THEORY

Consider a two-dimensional circularly symmetric signal $p(x,y)$ which has a circularly symmetric Fourier transform. Using the relationship $r = (x^2+y^2)^{1/2}$, the signal can be expressed in terms of the two-dimensional inverse Fourier transform, or equivalently, in terms of the Hankel transform as

$$p(r) = \int_0^\infty g(k_r)J_0(k_r r)k_r dk_r \quad (9)$$

where $J_0(\cdot)$ is the zeroth-order Bessel function. Since the Hankel transform is its own inverse, it follows that

$$g(k_r) = \int_0^\infty p(r)J_0(k_r r)r dr \quad (10)$$

These equations imply that $p(r)$ is an even function of r and that $g(k_r)$ is an even function of k_r.

Using the relationship between Bessel functions and Hankel functions, it is also possible to write a two-sided version of the Hankel transform in equation (9) as

$$p(r) \equiv \mathcal{H}^{-1}\{g(k_r)\}$$
$$= \frac{1}{2}\int_0^\infty g(k_r)H_0^{(1)}(k_r r)k_r dk_r + \frac{1}{2}\int_0^\infty g(k_r)H_0^{(2)}(k_r r)k_r dk_r \quad (11)$$

The unilateral version of this transform can be written as

$$p_u(r) \equiv \mathcal{H}_u^{-1}\{g(k_r)\} = \frac{1}{2}\int_0^\infty g(k_r)H_0^{(1)}(k_r r)k_r dk_r \quad (12)$$

and is analogous to the unilateral inverse Fourier transform in equation (2).

In some wave propagation problems, $p(r)$ is approximated by retaining only the first integral in equation (11), and thus

$$p(r) \sim p_u(r) \quad r > 0 \quad (13)$$

This approximation can be informally argued based on the outgoing nature of the propagating field, and can be more formally justified using a contour deformation argument. The approximation is an important component in a number of wave propagation synthetic data-generation methods, such as the Fast-Field-Program (FFP)[2]. However, it has not previously been recognized that equations (12) and (13) also imply an approximate real-part sufficiency condition for $p(r)$. Because of its important properties, and its relationship to the Hilbert and Hankel transforms, we refer to equation (12) as the ***Hilbert-Hankel transform.*** To the extent that the approximation in equation (13) is valid, it can be shown [1] that there is an inverse relationship between the Hilbert-Hankel transform and the complex Hankel transform[3].

To derive the real-part sufficiency condition, we use the fact that $H_0^{(1)}(k_r r) = J_0(k_r r) + jY_0(k_r r)$, where $Y_0(\cdot)$ is a zeroth-order Bessel function of the second kind. Substituting this expression in equation (12), the condition in equation (13) implies that

$$\frac{1}{2}\int_0^\infty g(k_r)J_0(k_r r)k_r dk_r \sim \frac{j}{2}\int_0^\infty g(k_r)Y_0(k_r r)k_r dk_r \quad (14)$$

By equating the real and imaginary parts on both sides of this expression, it can be shown that for $r > 0$

$$Re[p(r)] \sim -\int_0^\infty Y_0(k_r r)k_r dk_r \int_0^\infty Im[p(r)]J_0(k_r r)r dr \quad (15)$$

$$Im[p(r)] \sim \int_0^\infty Y_0(k_r r)k_r dk_r \int_0^\infty Re[p(r)]J_0(k_r r)r dr \quad (16)$$

RECONSTRUCTION ALGORITHM

It is also possible to use an additional approximation as a means for developing an efficient algorithm to reconstruct $Re[p(r)]$ from $Im[p(r)]$, or vice versa. To do this, we substitute an asymptotic expansion for $H_0^{(1)}(k_r r)$ in equation (12) yielding

$$p(r)u(r) \sim \frac{1}{(2\pi)^{1/2}}\int_0^\infty g(k_r)\frac{e^{j(k_r r-\pi/4)}}{(k_r r)^{1/2}}k_r dk_r \quad (17)$$

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so that

$$p(r)r^{1/2}u(r) \sim \frac{1}{(2\pi)^{1/2}} \int_0^\infty \tilde{g}(k_r)e^{jk_r r}dk_r \qquad (18)$$

where $\tilde{g}(k_r) = g(k_r)k_r^{1/2}e^{-j\pi/4}$. From this, it can be seen that $p(r)r^{1/2}u(r)$ is approximately analytic, and thus its real and imaginary parts are related via the Hilbert transform. In the reconstruction algorithm, a single component of $p(r)$ is multiplied by $r^{1/2}$, the Fourier transform is computed, the spectrum is multiplied by $2u(k_r)$, an inverse Fourier transform is computed, and the result is divided by $r^{1/2}$ [4]. The resultant signal is complex-valued, and consists of the original and the reconstructed quadrature components.

EXAMPLES

We have applied the reconstruction algorithm to a 220 Hz deep water acoustic field, generated synthetically to represent a realistic ocean environment, and to a 140 Hz shallow water acoustic field, collected experimentally[5] in Nantucket Sound in June 1984. The magnitude of the original deep water field, computed using the Hankel transform of a realistic Green's function[1], is shown in Figure 1a. The real component of the field was set to zero and then reconstructed. The magnitude of the reconstructed field is shown in Figure 1b and is seen to compare compare closely with the original magnitude. The magnitude of the experimental shallow water acoustic field is shown in Figure 2a. The real component of the field was set to zero and then reconstructed. The magnitude of the reconstructed field is shown in Figure 2b and its agreement with the original field magnitude is again quite good. Similar agreement between the phase of the original field and the reconstructed field was observed for these two examples.

SUMMARY

It has been shown that under the condition that the causal portion of an even signal is approximated by the unilateral inverse Fourier transform, there exists an approximate real-part sufficiency condition for the signal. Additionally, if the causal portion of a circularly symmetric two-dimensional signal is approximated by the Hilbert-Hankel transform, there exists an approximate real-part sufficiency condition. An additional asymptotic expansion was used to develop a two-dimensional reconstruction algorithm which was applied to synthetic and experimental acoustic fields.

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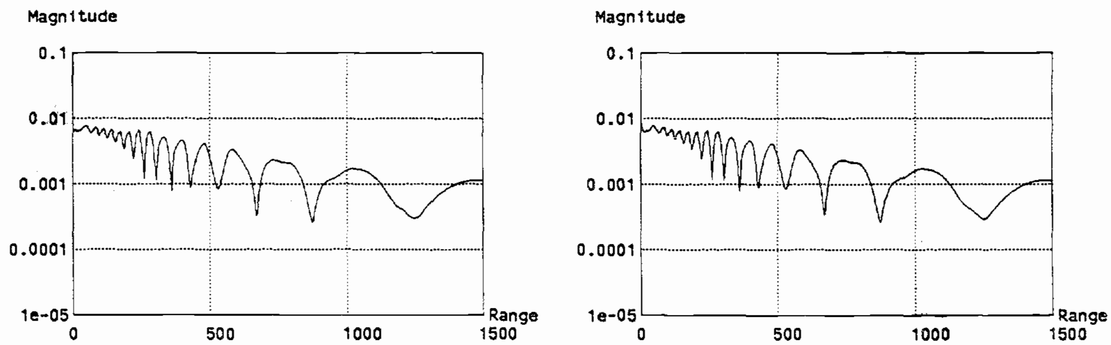
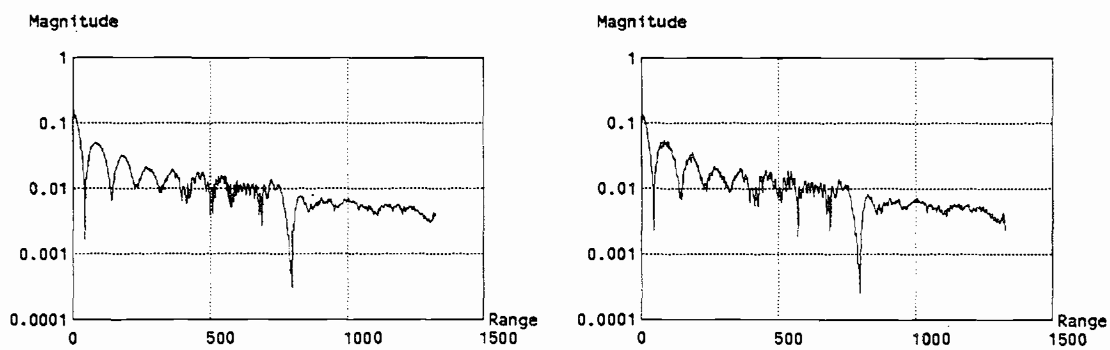


a) Original

b) Reconstructed

Figure 1: Synthetic deep water acoustic field magnitude as a function of range (m).



a) Original

b) Reconstructed

Figure 2: Experimental shallow water acoustic field magnitude as a function of range (m).