RECONSTRUCTION OF TWO-DIMENSIONAL SIGNALS FROM THRESHOLD CROSSINGS*

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ABSTRACT

In this paper, we present new results on the reconstruction of two-dimensional signals from zero crossing or threshold crossing information. These results follow from our previous work on the reconstruction of two-dimensional signals from one bit of Fourier transform phase. Experimental results illustrating image reconstruction from threshold crossings are included.

I. INTRODUCTION

The importance of the zero crossings (or more generally, threshold crossings) of a signal has long been recognized in a number of different applications and types of problems. Experiments in speech processing have shown that speech with only the zero crossing information preserved (hard-clipped speech) retains much of the intelligibility of the original speech [1]. A wide variety of papers in image processing and vision stress the importance of the information contained in the edges of objects in classifying and identifying images [2]. There are also a variety of other types of applications in which the threshold crossings of a signal are available and it is necessary or desirable to recover the original signal. One such application occurs when a signal is clipped or otherwise distorted in such a way as to preserve the zero crossing or level crossing information, and it is desired to recover the original signal. Another application occurs in some design problems where one might want to specify a filter response [3] or antenna pattern [4] in terms of zero crossing or null points (such as for interpolation) and derive the remainder of the response from these.

Although a considerable amount of research has been devoted to the problem of reconstruction of one-dimensional signals from zero crossings (see [5] for a survey and references), very little work has been reported on the corresponding two-dimensional problem. Logan's results [6] on the unique specification of one-dimensional bandpass signals with zero crossings have been extended to two dimensions in [2] and [7]. These extensions essentially require a one-dimensional signal derived from the original two-dimensional signal to satisfy the constraints of Logan's theorem.

In this paper, we present a more general set of conditions stated directly on the original signal which permit a bandlimited (but not necessarily bandpass) signal to be uniquely specified with zero crossings or with crossings of an arbitrary threshold. The result is much less restrictive and appears to be more broadly applicable than two-dimensional extensions of Logan's theorem. We shall develop this result for periodic signals in a style similar to that used to develop previous results [8, 9] on the unique specification of two-dimensional discrete-time sequences with the sign of the real part of the Fourier transform (also referred to as one bit of Fourier transform phase). The extension of the result presented in this paper to nonperiodic signals is developed in [10].

In the next section, we present the new theoretical results mentioned above on the reconstruction of two-dimensional signals from zero crossings or threshold crossings. Experimental results illustrating example images reconstructed from threshold crossings are given in section 3.

II. THEORETICAL RESULTS

Loosely speaking, the representation of a signal in terms of zero crossings can be thought of as a form of nonuniform sampling, with each zero crossing representing one sample. Most one-dimensional bandlimited signals are not uniquely determined by zero crossings since the average rate of zero crossings is not guaranteed to be sufficiently high. Logan's condition [6] requires signals to be bandpass with a bandwidth of less than one octave so that the number of zero crossings (or the rate of zero crossings) is in some sense consistent with the amount of information

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or bandwidth in the signal. In two dimensions, in contrast to one dimension, the "zero crossings" (sign changes) of a signal are contours and thus each zero crossing contour corresponds to an infinite number of samples of the signal. Thus it is reasonable to suggest that a two-dimensional signal may be specified with zero crossings under more general conditions than those required for a one-dimensional signal.

This is in fact true, and in this section, we shall present a new result on the unique specification of band-limited, periodic, two-dimensional signals from zero crossings. This result can be obtained in a straight-forward way by interchanging the roles of the signal and transform domains in the results on reconstruction from one bit of Fourier transform phase. Specifically, if a continuous-time signal corresponds to the Fourier transform of a finite-length discrete-time sequence, and if this finite-length sequence satisfies the conditions of any of the results in [8] or [9], then the signal is uniquely specified to within a scale factor by its zero crossings. This approach is developed in detail in [9]. An alternate approach is to develop this result directly by expressing the signal as a polynomial in a Fourier series representation and applying the same well-known results in algebraic geometry used in [8] and [9]. This latter approach shall be followed here. We shall present a brief sketch of the argument to give the basic idea behind the proof but omit the details.

Since the argument depends primarily upon a result from algebraic geometry, we will first state this result without proof:

> **Theorem 1** [11,12]. If $X(z_1, z_2)$ and $Y(z_1, z_2)$ are two-dimensional polynomials of degrees r and s with no common factors of degree > 0, then there are at most rs distinct pairs (z_1, z_2) where:
>
> $$X(z_1, z_2) = 0 \quad \text{and} \quad Y(z_1, z_2) = 0 \tag{1}$$

In this theorem, the degree of a polynomial in two variables is defined in terms of the sum of the degrees in each variable (for each term), that is, the degree of a two-dimensional polynomial $p(x,y)$ is equivalent to the degree of the one-dimensional polynomial $p(x,x)$. The rs distinct pairs (z_1, z_2) described in this theorem consist of rs points anywhere in the complex (z_1, z_2)-plane. Essentially, Theorem 1 places an upper bound on the number of points where two two-dimensional polynomials can both be zero if they do not have a common factor.

To see how this result applies to the problem of unique specification of two-dimensional signals with zero crossings, consider a real, band-limited, continuous-time, periodic signal $f(x,y)$ with periods T_1 and T_2 in the x- and y- directions, respectively. We can express $f(x,y)$ as a polynomial using the Fourier series representation:

$$f(x,y) = \sum_{n_1}\sum_{n_2} F(n_1, n_2)\,(e^{j\frac{2\pi x}{T_1}})^{n_1}(e^{j\frac{2\pi y}{T_2}})^{n_2} \tag{2}$$

where the sums are finite since $f(x,y)$ is bandlimited. Then if another signal $g(x,y)$ has the same zero crossings as $f(x,y)$ and these zero crossings are contours consisting of an infinite number of points, $f(x,y)$ and $g(x,y)$ must have a common factor. If furthermore we know that $f(x,y)$ and $g(x,y)$ are irreducible when expressed as polynomials as in equation (2), then they must be equal to within a scale factor. Our result can be stated as follows:

> **Theorem 2.** Let $f(x,y)$ and $g(x,y)$ be real, two-dimensional, doubly-periodic, continuous, band-limited functions with $\text{sign} f(x,y) = \text{sign}\, g(x,y)$, where $f(x,y)$ takes on both positive and negative values. If $f(x,y)$ and $g(x,y)$ are nonfactorable when expressed as polynomials in the Fourier series representation (2), then $f(x,y) = cg(x,y)$.

Although we have stated a specific set of conditions under which signals are uniquely determined by zero crossings, it is possible to extend this result in a variety of different ways. Since many two-dimensional signals encountered in practice are not periodic but have finite support, we will first modify Theorem 2 so that it applies to these signals. Consider the case where $f(x,y)$ is a finite segment of a periodic signal satisfying the constraints of Theorem 2. For example, if $f(x,y)$ represents one period of a band-limited periodic function $\tilde{f}(x,y)$:

$$\tilde{f}(x,y) = \sum_{n_1}\sum_{n_2} f(x + n_1T_1, y + n_2T_2) \tag{3}$$

then it is possible to recover $f(x,y)$ from its zero crossings provided that $\tilde{f}(x,y)$ satisfies the constraints of Theorem 2, even though $f(x,y)$ itself is not band-limited. More generally, it is not necessary for the duration of $f(x,y)$ to be equal to one period of the corresponding periodic function. Thus, $f(x,y)$ can represent a finite segment of a variety of different periodic functions. In order for $f(x,y)$ to be uniquely specified by its zero crossings, we only need one periodic function containing $f(x,y)$ to be band-limited.

It is also possible to generalize these theorems to allow crossings of an arbitrary threshold rather than just zero crossings. Specifically, if the signal $f_2(x,y) = f(x,y) - a$ satisfies the constraints of Theorem 2 or its extensions, then $f(x,y)$ is uniquely specified by the set of points where it crosses the threshold a. This result is important in applications such as image processing where signals are positive everywhere and thus have no zero crossings. It is also worthwhile to note that any threshold can be used as long as the signal takes on values both above and below the threshold so that at least one "threshold crossing" contour will exist. Some examples are included in the next section to illustrate this point.

It is also worthwhile to note that all of the results presented above require the signals to be irreducible. It has been shown [13] that almost all two-dimensional polynomials are irreducible, and thus any signal encountered in practice is quite likely to satisfy the irreducibility constaint if it is bandlimited and can be expressed by a finite-order Fourier series. The irreducibility constraint can also be replaced with a constraint on each factor. The details of this and other extensions can be found in [9].

III. EXPERIMENTAL RESULTS

Having established theoretical results on reconstruction from zero crossings, it is now worthwhile to examine the problem of recovering an actual signal from zero crossing information alone. The method we will use here is to solve a set of linear equations for the Fourier series coefficients of the signal. Our primary purpose in this section is to demonstrate the feasibility of recovering signals from zero crossings and not to suggest that the linear equation method is the best or the only approach. A number of different algorithms are possible and the further investigation and evaluation of these is a subject for future research.

A set of linear equations can be written by considering each zero crossing point to be one linear constraint on the Fourier series coefficients, i.e.,

$$\sum_{n_1} \sum_{n_2} F[n_1, n_2]\, e^{j\frac{2\pi x_i n_1}{T_1}}\, e^{j\frac{2\pi y_i n_2}{T_2}} = 0 \qquad (4)$$

where each equation uses a different pair of points (x_i, y_i) for which the equality is known to hold. (We substitute $F[0,0] = 1$ in order to obtain a non-zero solution.) Although this method is very sensitive to numerical errors in the values of (x_i, y_i), if the values of x and y are known exactly (to within the limits of double precision) and if the number of equations used is greater than the number of unknowns and a least-squares solution is obtained, results indistinguishable from the original signal have been achieved.

An example of the results obtained with this method is shown in Figure 1, where (a) shows the original image, (b) shows the threshold crossings, that is, the original image quantized to one bit, and (c) shows the recovered image. This image was recovered by first determining the zero crossing locations to 16 digits of accuracy, then finding the least squares solution to the set of equations described above, and then using an inverse FFT to obtain the signal from its Fourier series coefficients. In this figure, 574 equations (in 454 unknowns) were used and the resulting image has a normalized rms error (rms error/rms signal) of 0.000067.

In this example, the threshold was arbitrarily chosen to be somewhere near the mean value of intensity of the original. Other choices of threshold are possible since theoretically, it is possible to use any threshold as long as the signal crosses the chosen threshold at some point. However, the choice of threshold does effect the numerical sensitivity of the procedure and the number of equations which must be used in practice. For example, Figure 2 shows another example where a different intensity level was chosen as the threshold. In this figure, (a) shows the threshold crossings of Figure 1(a) using this new threshold, and (b) shows the recovered image. In Figure 2, 1012 equations were required and the resulting image has a normalized error of 0.0021.

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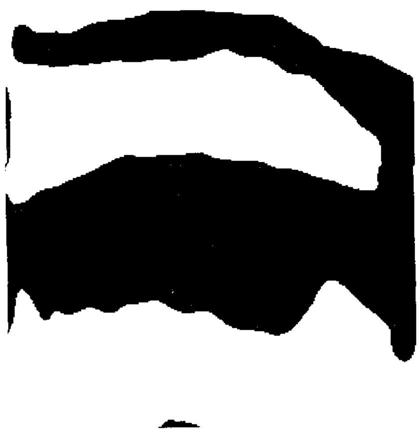
(a) original image



(b) image showing threshold crossings of (a)

(c) image recovered from (b)

Figure 1. Reconstruction from Threshold Crossings

(a) image showing a different threshold

(b) image recovered from (a)

Figure 2. Reconstruction with Different Threshold