Iterative procedures for signal reconstruction from Fourier transform phase

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Abstract. Recently, a set of conditions has been developed under which a sequence is uniquely specified by the phase or samples of the phase of its Fourier transform. These conditions are distinctly different from the minimum or maximum phase requirement and are applicable to both one-dimensional and multi-dimensional sequences. Under the specified conditions, several numerical algorithms have been developed to reconstruct a sequence from its phase. In this paper, we review the recent theoretical results pertaining to the phase-only reconstruction problem, and we discuss in detail two iterative numerical algorithms for performing the reconstrucction.

Keywords: signal reconstruction; phase; Fourier transform.

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CONTENTS

1. INTRODUCTION

Under a variety of conditions a signal can be completely reconstructed from partial information about its Fourier transform. For example, if a signal is known to be causal (i.e., zero for negative values of its argument), it can be exactly recovered from the real part or, except for its value at the origin, from the imaginary part of its Fourier transform. If it satisfies the minimum phase condition, it can also be exactly recovered from the magnitude or, to within a scale factor, from the phase of its Fourier transform. Other possible conditions have been explored under which the signal reconstruction can be accomplished from partial information.

Reconstruction of a signal from such partial information is important and useful in a broad set of important practical applications. For example, in some cases of optical image processing or in measurement of diffraction patterns, only spectral magnitude information can be recorded or is available, and thus it is of interest to recover a signal from spectral magnitude information alone.[1] Related problems are the reconstruction of a signal from intensity measurements in two domains[2-4] and the reconstruction of a signal when it is known only over a specified band in the frequency domain and a specified interval in the time domain.[5] In other situations, either the spectral magnitude or phase may be badly distorted, and restoration must rely on the undistorted component. For example, in the class of problems referred to as blind deconvolution,[6] a desired signal is to be recovered from an observation which is the convolution of the desired signal with some unknown distorting signal. Since little is usually known about either the desired signal or the distorting signal, deconvolution of the two signals is generally a very difficult problem. However, in the special case in which the distorting signal is known to have a phase which is identically zero, the phases of the observed signal and the desired signal are identical. In such situations, it may be of interest to consider signal reconstruction from phase information alone. It is also likely that signal reconstruction from only the Fourier transform phase can be useful in the estimation of the frequency response of a linear time-invariant system if, for example, the symmetry of an input to the system can be controlled.

In this paper we focus specifically on the problem of reconstructing a signal from Fourier transform phase information alone. In general, of course, a sequence is not uniquely defined by its phase,* as is illustrated by the observation that a sequence convolved with any zero-phase sequence will produce another sequence with the same phase. Thus, without some assumptions about the sequence, the phase may, at best, uniquely specify a sequence only to within an arbitrary zero-phase factor. One well-known set of conditions for reconstruction from phase is the minimum phase condition. Recently,[7] we have developed new conditions under which a sequence is uniquely defined by the phase of its Fourier transform. These conditions are applicable to both one-dimensional (1-D) and multidimensional (M-D) sequences. Furthermore, we have developed several numerical algorithms to reconstruct a sequence from its associated phase under the specified conditions. In this paper we review the recent theoretical results pertaining to the phase-only reconstruction problem, and we discuss in detail two iterative numerical algorithms for performing the reconstruction.

2. SIGNAL RECONSTRUCTION FROM FOURIER TRANSFORM PHASE

As mentioned in the introduction, a sequence is not uniquely defined by the phase of its Fourier transform without some additional knowledge about the sequence. In this section, we summarize four recently developed theorems embodying conditions under which a finite length signal is recoverable from its associated phase. Justification of the theorems is presented in Ref. 7.

Theorem 1: A 1-D sequence which is finite in length and has a

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*Throughout this paper reference to the phase associated with a sequence should be explicitly interpreted as the phase of the Fourier transform of the sequence.

z-transform with no zeros in conjugate reciprocal pairs or on the unit circle is uniquely specified to within a scaling factor by the phase of its Fourier transform (or by the tangent of the phase). The condition which excludes zeros from the unit circle is made only for convenience. The condition which excludes zeros in conjugate reciprocal pairs, however, is necessary to eliminate the possible ambiguity due to zero-phase components. This theorem is also applicable to all-pole sequences since the convolutional inverses of these sequences are finite in length.

Although Theorem 1 is formally stated for 1-D sequences, an extension to M-D sequences has been accomplished by using the projection-slice theorem.[8] This theorem establishes the result that an M-D sequence having a rational z-transform may be mapped into a 1-D sequence (projection) by means of an invertible transformation. For example, a 2-D finite extent sequence with n rows and m columns can be mapped into a 1-D finite extent sequence of length nm by concatenating the columns, which represents one particular projection of the sequence. In general, the transformation in which an M-D sequence is represented by a 1-D projection has the property that the phase of the projection is equal to a slice of the phase of the M-D sequence, and thus, in particular, the phase of the projection is uniquely defined by the phase of the M-D sequence. Consequently, the multidimensional phase-only problem can be mapped into a one-dimensional phase-only problem, and the phase-only reconstruction theorem for 1-D sequences may be used.

The approach of transforming M-D sequences into 1-D projections provides at least a partial solution to the multidimensional phase-only problem. However, this approach circumvents the fundamental issues involved in multidimensional phase-only signal reconstruction. For example, it imposes constraints on a projection of an M-D sequence rather than directly on the M-D sequence. In addition, although it may not be possible to perform a phase-only reconstruction of an M-D sequence from a particular projection, this does not preclude the possibility that there exists another projection or mapping for which the reconstruction is possible. Therefore, with this approach it is difficult to determine which multidimensional sequences may be reconstructed from their phase. However, Theorem 1 can be extended to M-D sequences through the following theorem:[9]

Theorem 2: An M-D sequence which has finite support and a z-transform with no symmetric factors* is uniquely specified to within a scale factor by the phase (or tangent of the phase) of its M-dimensional Fourier transform.

Clearly, Theorem 1 is a special case of this theorem. However, the proof of the general M-dimensional theorem is more abstract than that required in the one-dimensional case.

Although the phase-only reconstruction theorems specify a set of conditions under which a sequence is uniquely specified to within a scale factor by the phase of its Fourier transform, it is assumed in these theorems that the phase is known for all frequencies. Since any practical algorithm for reconstructing a sequence from the phase will base the reconstruction on only a finite set of samples of the phase, the following theorem extends Theorem 1 to consider the uniqueness of a sequence based only on phase samples.

Theorem 3: A sequence which is known to be zero outside the interval $0 \leq n \leq (N\text{-}1)$ and which has a z-transform with no zeros on the unit circle or in conjugate reciprocal pairs is uniquely specified to within a scale factory by (N-1) samples of the phase of its Fourier transform (or the tangent of the phase) at distinct frequencies in the interval $0 < \omega < \pi$.

This theorem forms the basis for demonstrating the existence and uniqueness of solutions to the signal reconstruction algorithms which are described in the next section. The extension of this theorem to multidimensional sequences is as follows:

Theorem 4: An M-D sequence which is known to be zero outside the region* $\mathbf{0} \leq \mathbf{n} < \mathbf{N}$ and a z-transform with no symmetric factors is uniquely specified to within a scale factor by the phase of its **M**-point discrete Fourier transform, provided $\mathbf{M} > 2(\mathbf{N}\text{-}1)$.

*A symmetric factor is defined to be of the form

$$F(\mathbf{z}) = \pm \mathbf{z}^{\mathbf{k}} F(\mathbf{z}^{-1})$$

for some integer-valued vector **k**.

*If k and ℓ are two vectors of length m then $\mathbf{k} < \ell$ means that $k_i < \ell_i$ for $i = 1, \ldots, m$.

3. ALGORITHMS FOR SIGNAL RECONSTRUCTION FROM PHASE

In the previous section, we stated that a finite duration signal which has no zeros on the unit circle or in conjugate reciprocal pairs is uniquely specified by samples of its associated phase function. In this section, we describe several numerical algorithms to reconstruct a one-dimensional signal from its phase function when the signal satisfies these constraints. The extension of these algorithms to the multidimensional case is straightforward, and the details may be found in Ref. 9. In describing these algorithms, x[n] is used to denote a 1-D sequence, and $\theta_x(\omega)$ is used to denote the phase associated with x[n]. The sequence x[n] is assumed to be zero outside the interval $0 \leq n \leq N\text{-}1$ with $x[0] \neq 0$ and to have no zeros on the unit circle or in conjugate reciprocal pairs. The additional constraint that $x[0] \neq 0$ is not necessary, but its inclusion is not overly restrictive in practice and simplifies the algorithms.

One algorithm for reconstructing x[n] to within a scale factor from its phase $\theta_x(\omega)$ involves solving a set of linear equations and leads to a closed form solution. Specifically, from the definition of $\theta_x(\omega)$, it can be shown[7] that x[n] satisfies the equation

$$\sum_{n=1}^{N-1} x[n]\sin[\theta_x(\omega) + n\omega] = -x[0]\sin\theta_x(\omega)\,. \quad (1)$$

When sampled at (N-1) distinct frequencies in the interval $0 < \omega < \pi$, N-1 linear equations are obtained for the unknowns x[n]. It can be shown[7] that given x[0], these linear equations can be solved to uniquely determine x[n] for $1 \leq n \leq N\text{-}1$, and this unique solution is the desired one.

Even though the algorithm corresponding to solving the above set of linear equations may be used in principle to recover a signal from its phase function, its application in practice may be quite limited, if N is large, due to the potential computational problem inherent in solving a large set of linear equations. For example, if an image of 256×256 pixels is mapped to a one-dimensional finite length sequence by concatenating columns, the algorithm requires solving a set of 2^{16}-1 linear equations. Solving such a large set of equations will generally lead to numerical instability and severe round-off errors. As an alternative we consider two iterative algorithms for carrying out the reconstruction, in which the estimate of x[n] is improved in each iteration. The first algorithm is in a form similar to the Gerchberg-Saxton algorithm[2] and iterative algorithms developed by Quatieri.[10] The second algorithm is a revision of the first algorithm which noticeably improves its convergence characteristics.

3.1. Iterative algorithm A

This algorithm involves repeated transformation between the time and frequency domains in which the known constraints are imposed in each domain. Specifically, we denote the M point discrete Fourier transform (DFT) of x[n] by

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi}{M}K} = |X(k)|e^{j\theta_x(k)}\,. \quad (2)$$

With $M \geq 2N$, we begin the iterative procedure with an initial guess of the unknown DFT magnitude $|X_0(k)|$. From $|X_0(k)|$ and the given phase samples $\theta_x(k)$, the first estimate of x[n], which we denote by $x_1[n]$, is formed as

$$x_1[n] = \mathrm{IDFT}[|X_0(k)|e^{j\theta_x(k)}], \tag{3}$$

where IDFT denotes the M point inverse discrete Fourier transform operation. Since an M point DFT and IDFT with $M \geq 2N$ is used in the above procedure, $x_1[n]$ is an M point sequence which is generally non-zero for $N \leq n \leq M-1$. From $x_1[n]$, we then form another sequence $y_1[n]$ by imposing the constraint that the last (M-N) points be zero and that the first point be equal to some arbitrary constant, i.e.,

$$y_1[n] = \begin{cases} x_1[n] & \text{for } 0 < n \leq N-1 \\ \beta & \text{for } n = 0 \\ 0 & \text{for } N \leq n \leq M-1 \end{cases} . \tag{4}$$

The magnitude $|Y_1(k)|$ of the M point DFT of $y_1[n]$ is then considered to be a new estimate of $|X(k)|$ and a new estimate of $x[n]$ is formed as

$$x_2[n] = \mathrm{IDFT}[|Y_1(k)|e^{j\theta_x(k)}], \tag{5}$$

i.e., the known phase samples $\theta_x(k)$ are substituted in place of the phase associated with $y_1[n]$. Equations (3), (4), and (5) complete one iteration, and repetitive application of this procedure defines the iteration. This iterative algorithm is illustrated in Fig. 1.

It has recently been shown theoretically[11] that this iterative algorithm always leads to a converging solution, provided the DFT length M is greater than 2N-1, and that the unknown sequence satisfies the constraints noted above. Furthermore, consistent with this theoretical result, it has been empirically observed that the algorithm always converges. Several examples of this iterative algorithm will be shown in Section 4.

Since the DFT and IDFT are the major computational elements of the iteration, the algorithm does not have the same numerical instability or severe round-off errors for relatively large N that would occur in solving Eq. (1) through the use of a matrix inversion of size (N-1) × (N-1). However, as will be discussed in Section 4, the above iterative algorithm requires, in general, a large number of iterations before the converging solution is reached. Since each iteration requires one M point DFT and IDFT, considerable computation time can be saved by improving the convergence characteristics. In the next section, we consider a modification of the algorithm to provide more rapid convergence.

3.2. Iterative algorithm B

The iterative algorithm described above may be represented mathematically as

$$\mathbf{x}_{p+1} = T\mathbf{x}_p, \tag{6}$$

where $\mathbf{x}_p^t = [x_p[0], x_p[1], \ldots, x_p[M]]$ and $\mathbf{x}_{p+1}^t = [x_{p+1}[0], x_{p+1}[1], \ldots, x_{p+1}[M]]$ correspond to the vectors that represent the estimates of the unknown vector $\mathbf{x}^t = [x[0], x[1], \ldots, x[M]]$ after p and p+1 iterations, respectively, and where T is a nonlinear operator which corresponds to the combination of the time-limiting and phase-substitution operators in Fig. 1. Motivated by the various relaxation techniques developed for iterative algorithms,[12] consider the vector $\mathbf{r}_p$ defined by

$$\mathbf{r}_p = \mathbf{x}_{p+1} - \mathbf{x}_p = T\mathbf{x}_p - \mathbf{x}_p . \tag{7}$$

Now suppose that the iteration (6) is modified as follows:

$$\mathbf{x}_{p+1} = \mathbf{x}_p + \alpha_p \mathbf{r}_p, \tag{8}$$

where α_p is a scaler which will be referred to as the relaxation parameter and may be allowed to vary as a function of p. Using (7), an equivalent representation of the iteration (8) is

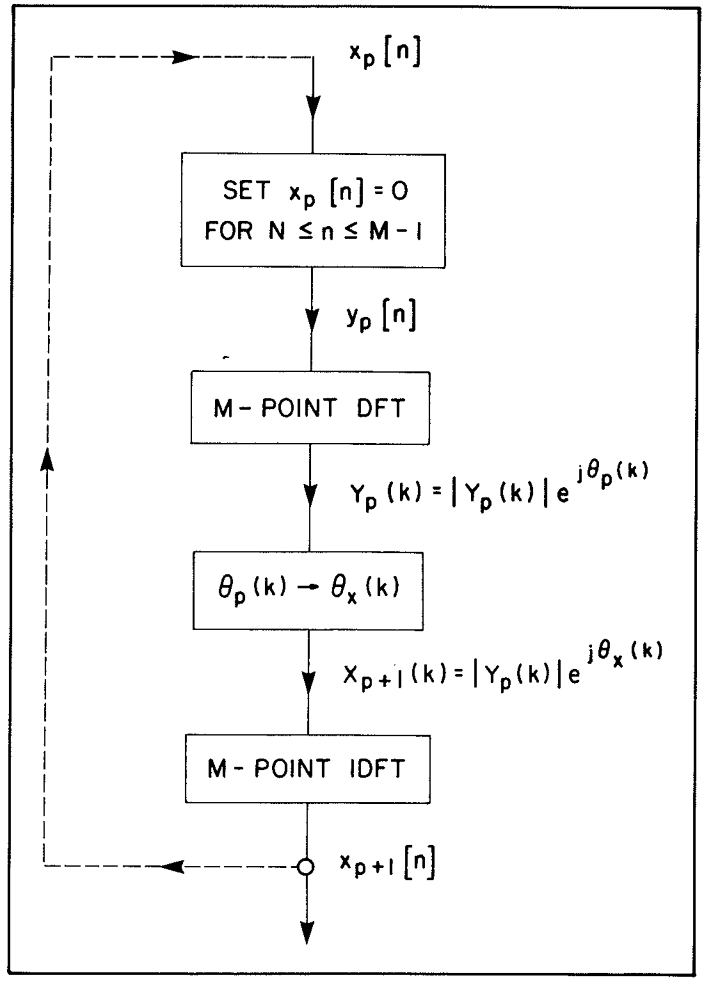

Fig. 1. Block diagram of the iterative algorithm for reconstructing a signal from its phase.

$$\mathbf{x}_{p+1} = (1-\alpha_p)\mathbf{x}_p + \alpha_p T\mathbf{x}_p \tag{9}$$

An important property of (9) is that the tangent of the M samples of the phase associated with $\mathbf{x}_{p+1}$ equals the tangent of the M samples of the phase associated with $\mathbf{x}$ for any choice of the relaxation parameter α_p. This follows from the observation that the Fourier transforms of both $\mathbf{x}_p$ and $T\mathbf{x}_p$ have the same M phase samples as those of $\mathbf{x}$ as a result of the definition of the operator T. Therefore, since $\mathbf{x}$ is uniquely specified by N-1 independent samples of the tangent of its associated phase, a convergent solution to (9) will, under the appropriate constraints, correspond to a scaled version of $\mathbf{x}$.

Several special cases of (9) are immediately apparent. If α_p is a fixed constant α_0, then (9) corresponds to the basic iteration (6) when $\alpha_0 = 1$, while $\alpha_0 = 0$ yields the trivial result $\mathbf{x}_{p+1} = \mathbf{x}_p$. Intermediate values of α_0, i.e., $0 < \alpha_0 < 1$, correspond to an under-relaxed version of (6).

A common limitation with iterations in the form of (9) is in the determination of the optimum value of the relaxation parameter α_p which maximizes the rate of convergence of the iteration. However, in the context of signal reconstruction from its phase, it is possible to derive a relatively simple method for computing the value of α_p which is optimum in a certain sense.[13] Specifically, consider partitioning (8) as follows:

$$\begin{bmatrix} \mathbf{x}_{p+1}^{(1)} \\ \text{------} \\ \mathbf{x}_{p+1}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{p}^{(1)} \\ \text{------} \\ \mathbf{x}_{p}^{(2)} \end{bmatrix} + \alpha_p \begin{bmatrix} \mathbf{r}_{p}^{(1)} \\ \text{------} \\ \mathbf{r}_{p}^{(2)} \end{bmatrix}, \tag{10}$$

where $x_{p+1}^{(2)}$, $x_p^{(2)}$, and $r_p^{(2)}$ are vectors of length (M-N) corresponding to the interval over which x (the desired vector) is known to be zero. Noting that a convergent solution is obtained if and only if $x_{p+1}^{(2)} = \mathbf{0}$, a reasonable approach for selecting α_p is to choose that value $\alpha_p = \hat{\alpha}_p$ which minimizes $\|x_{p+1}^{(2)}\|^2$, i.e.,

$$\left[\frac{d}{d\alpha_p} \|x_{p+1}^{(2)}\|^2 \right]_{\alpha_p = \hat{\alpha}_p} = 0 , \tag{11}$$

where $\|x_{p+1}^{(2)}\|^2$ is the square of the length of the vector $x_{p+1}^{(2)}$. Geometrically, $\hat{\alpha}_p$ defines that vector $x_{p+1}^{(2)}$ which is closest to the origin. Using (10), the solution to (11) is given by

$$\hat{\alpha}_p = -\frac{< x_p^{(2)}, r_p^{(2)} >}{\|r_p^{(2)}\|^2} , \tag{12}$$

where $< x_p^{(2)}, r_p^{(2)} >$ is the inner product of the vectors $x_p^{(2)}$ and $r_p^{(2)}$. Assuming that the DFT length used in the iteration is $M = 2N$, the number of multiplications required to compute $\hat{\alpha}_p$ is M, and the number of multiplications required to determine x_{p+1} in (10) is also M. Therefore, this approach requires an additional 2M multiplications per iteration over the basic iteration (6). Since the number of multiplications required for each iteration in (6) is on the order of $M\log_2 M$, if $M >> 1$ this additional computation is negligible. However, an important consideration in the implementation of (9) is the requirement for additional memory since two vectors of length M, namely, x_p and Tx_p, need to be stored.

An important practical and theoretical question concerns the conditions under which the relaxed algorithm (9) will converge. It may easily be shown that if α_p does not take on values outside the interval (0,1), then the iteration will always converge.[11] However, when α_p is defined by (12), there is no assurance that α_p will not take on values outside (0,1). In fact, using (12), it has been observed that although α_p always appears to assume non-negative values, values greater than 10 are not uncommon. Nevertheless, in all of the examples which have been considered, convergence of the iteration has always been achieved.

The iteration of Eq. (8) may be considered as a first-order acceleration of the basic iteration given by Eq. (6) since it incorporates one previous estimate, x_p, of $\mathbf{x}$ to modify the current estimate Tx_p. It is possible to generalize the iteration of Eq. (8) so that it incorporates more than one previous estimate to modify the current estimate Tx_p. Specifically, by expressing x_{p+1} as a linear combination of x_p and the differences between Tx_p and previous estimates of $\mathbf{x}$, it is straightforward to show that the tangent of the M phase samples of x_{p+1} equals the tangent of M phase samples of $\mathbf{x}$ for any choice of the relaxation parameters. In addition, the relaxation parameters can be obtained by generalization of Eq. (11). Even though it is expected that a higher order acceleration will improve the convergence rate, its implementation requires additional memory to store the previous estimates of $\mathbf{x}$.

Finally, it should be noted that the use of the adaptive relaxation (9) of the basic iteration (6) as well as its generalized form (14) is not limited to the above iterative algorithm. There are two properties, however, which permitted the development of the iterations (9) and (14). Therefore, any other iteration which has these same properties may be similarly extended. The first property is that a part of the unknown vector x is known *a priori.* For example, in the phase-only signal reconstruction problem, x[n] is known to be zero outside the interval $0 \leq n \leq N-1$ so that the last (M-N) components of $\mathbf{x}$ are known to be zero. This property allows for the explicit evaluation of the "optimum" relaxation parameter (12) or relaxation vector (17). The second property is that a linear combination of two or more estimates may be formed without affecting the constraints imposed by the iteration. In the above iteration, for example, linear combinations preserve the desired phase constraint. Another example[13] in which these two properties are satisfied so that a similar extension is possible is the band-limited extrapolation procedure proposed by Gerchberg[5] for obtaining super-resolution of images.

4. EXAMPLES

In this section, we illustrate an example in which an image is reconstructed iteratively from the phase of its Fourier transform. The example presented includes a reconstruction based both on the basic iteration (6) as well as the first-order acceleration technique (9).

Shown in Fig. 2(a) is an original image, 128 × 128 pixels in extent, which is to be reconstructed from its phase. Using a 256 × 256 point two-dimensional DFT, the phase-only representation of this image, which is obtained by setting the DFT magnitude equal to a constant, is shown in Fig. 2(b). With this phase-only image as the initial estimate in the iteration, the estimates obtained after 10, 20, 50, and 100 iterations are shown in Fig. 3 (each image has been appropriately scaled for display).

As has been discussed in Sec. 2, the motivation behind algorithm B is to provide more rapid convergence at the expense of a modest increase in computation at each iteration as well as an increase in memory requirements. In Figs. 4(a)–(d) are shown the images reconstructed from the phase of the image in Fig. 2 using algorithm B with the first-order acceleration after 5, 10, 20, and 30 iterations. The DFT length and initial estimate of $\mathbf{x}(n_1,n_2)$ used in generating

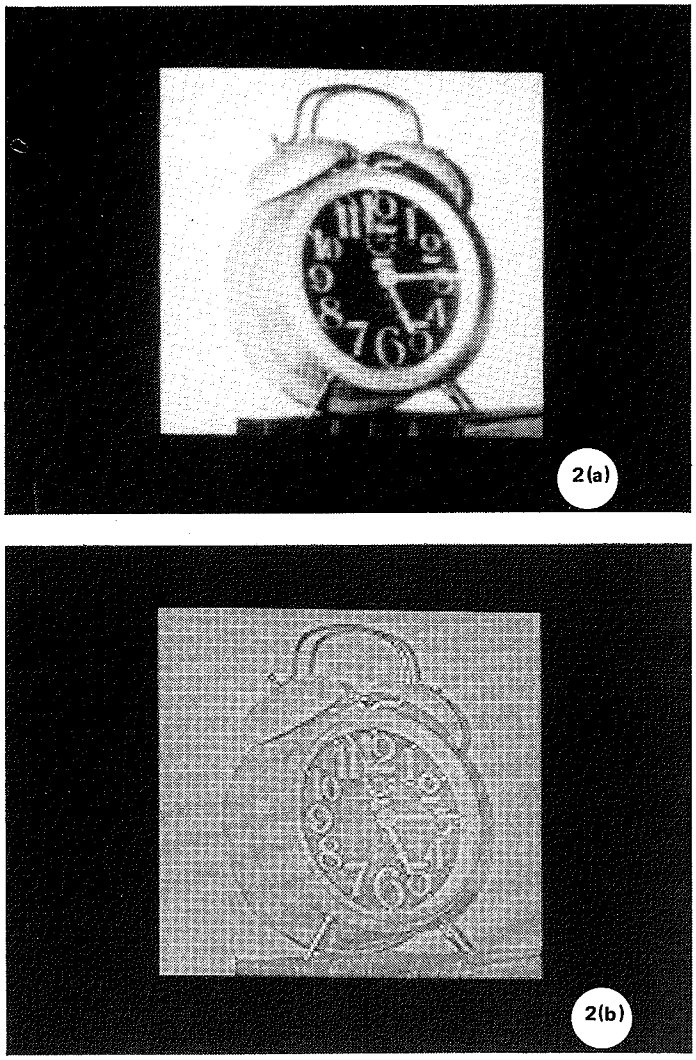

Fig. 2. Original image and its phase-only representation obtained by setting the Fourier transform magnitude equal to a constant. (a) original image; (b) phase-only image.

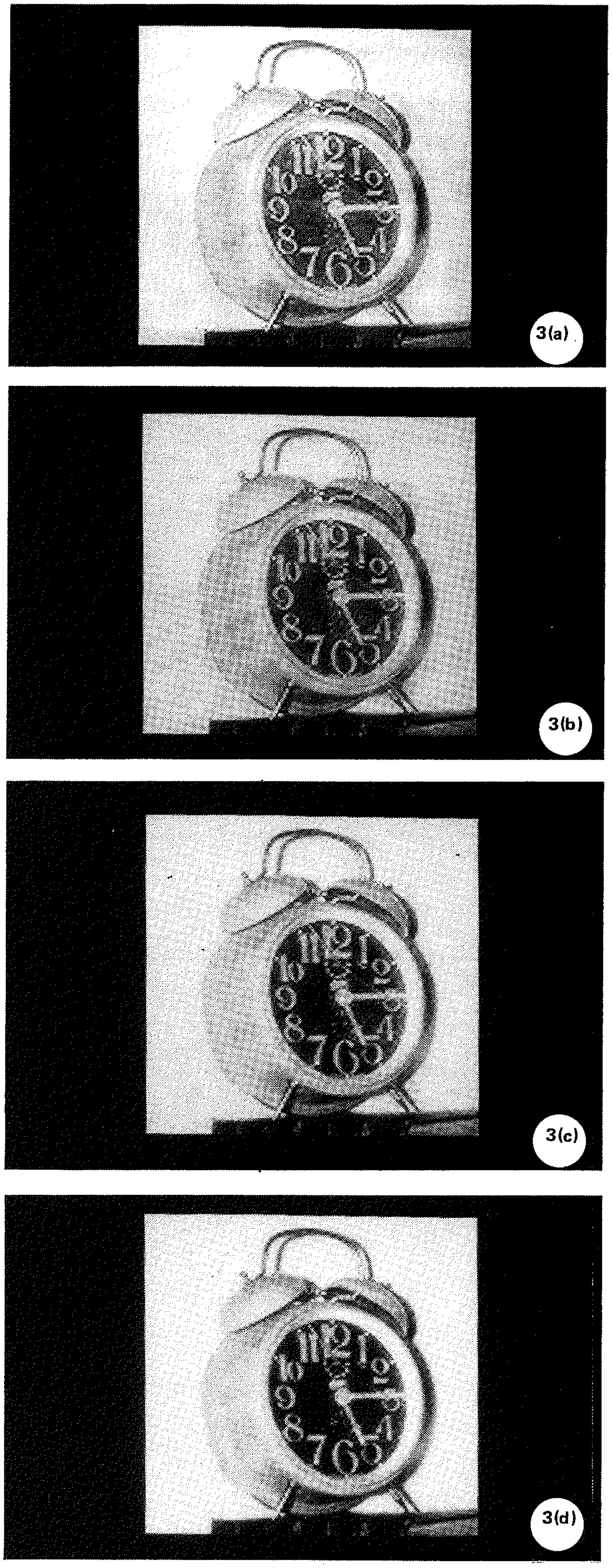

Fig. 3. Iterative reconstruction from phase. (a) 10 iterations; (b) 20 iterations; (c) 50 iterations; (d) 100 iterations.

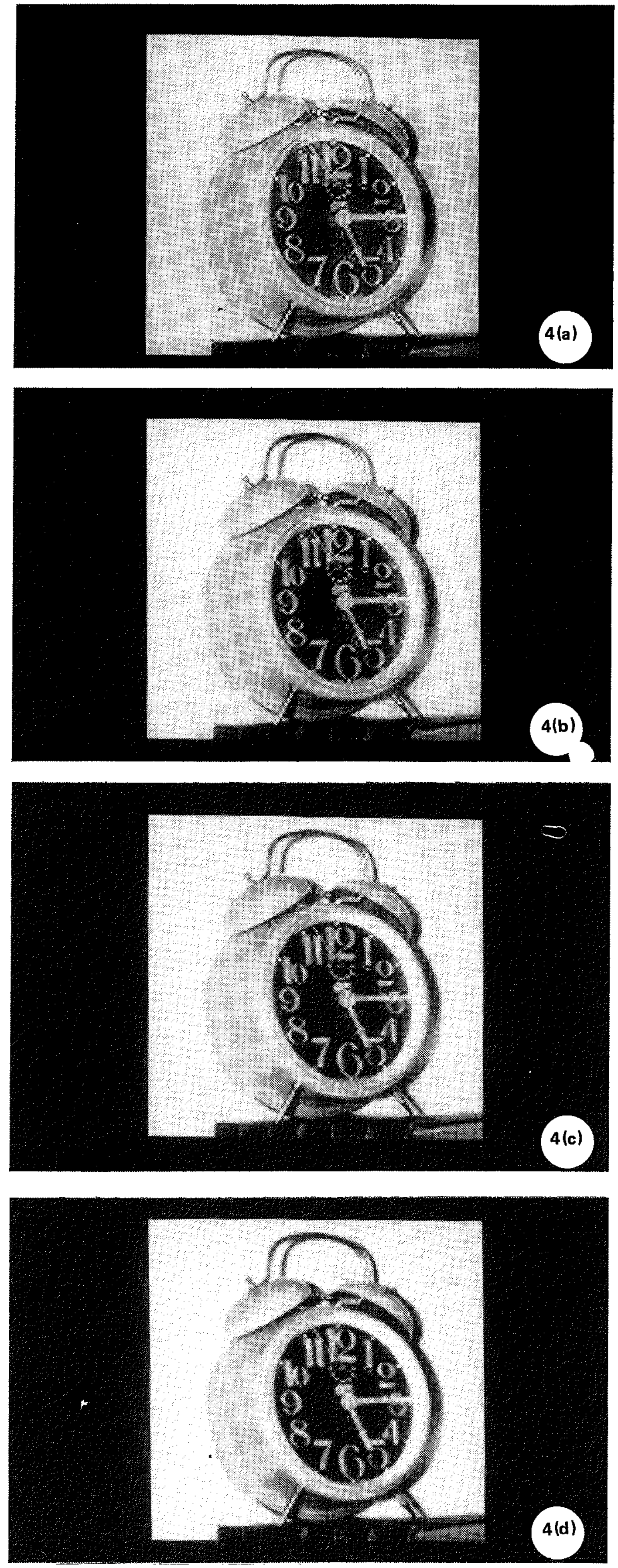

Fig. 4. Iterative reconstruction from phase using adaptive relaxation. (a) 5 iterations; (b) 10 iterations; (c) 20 iterations; (d) 30 iterations.

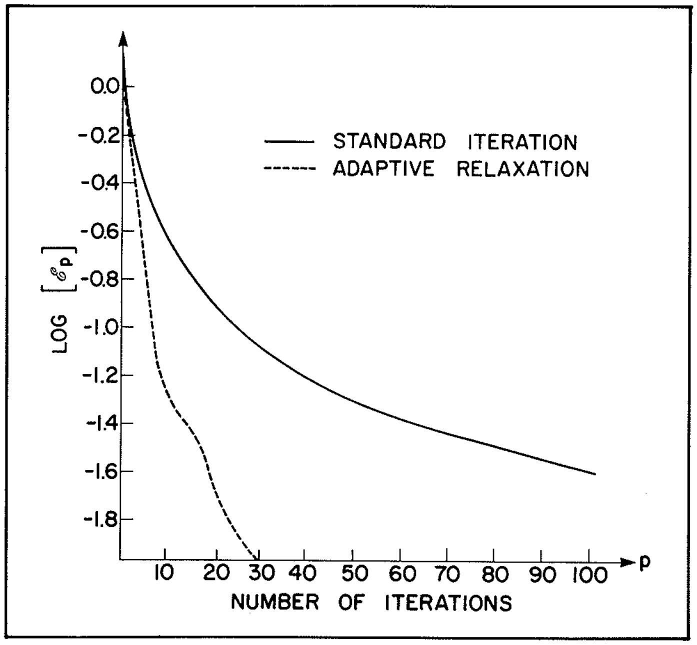

Fig. 5. Normalized mean square error versus the number of iterations for the standard iteration (algorithm A) and the adaptive relaxation algorithm (algorithm B).

these images are the same as in Fig. 3. Both visual and quantitative comparisons between Figs. 3 and 4 indicate that the number of iterations required in algorithm B to achieve approximately the same performance is significantly less than the number of iterations required in algorithm A. In particular, in Fig. 5 we have plotted the log of the normalized mean square error $\mathscr{E}_p$ as a function of p where $\mathscr{E}_p$ is defined as

$$\mathscr{E}_p = \frac{1}{N_1 N_2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left[\frac{x(n_1,n_2)}{\sigma_x} - \frac{x_p(n_1,n_2)}{\sigma_p} \right]^2, \quad (13)$$

where $N_1 = N_2 = 128$, and σ_x and σ_p are the standard deviations of $x(n_1,n_2)$ and $x_p(n_1,n_2)$, respectively. This error criterion was chosen since it is invariant to scaling of either $\mathbf{x}$ or $\mathbf{x}_p$. Note, in particular, that the error decreases much more rapidly with the adaptive relaxation algorithm B. A similar result has been obtained in various other examples.

5. ACKNOWLEDGMENTS

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