# Iterative Techniques for Minimum Phase Signal Reconstruction from Phase or Magnitude

THOMAS F. QUATIERI, JR., MEMBER, IEEE, AND ALAN V. OPPENHEIM, FELLOW, IEEE

*Abstract*—**In this paper, we develop iterative algorithms for reconstructing a minimum phase sequence from the phase or magnitude of its Fourier transform. These iterative solutions involve repeatedly imposing a causality constraint in the time domain and incorporating the known phase or magnitude function in the frequency domain. This approach is the basis of a new means of computing the Hilbert transform of the log-magnitude or phase of the Fourier transform of a minimum phase sequence which does not require phase unwrapping. Finally, we discuss the potential use of this iterative computation in determining samples of the unwrapped phase of a mixed phase sequence.**

## I. INTRODUCTION

UNDER certain conditions a signal can be reconstructed from a partial specification in the time domain, in the frequency domain, or in both domains. A minimum or maximum phase signal, in particular, can be recovered from the phase or magnitude of its Fourier transform [1]. The conventional reconstruction algorithm involves applying the Hilbert transform to the log-magnitude or phase of the Fourier transform to obtain the unknown component.

In this paper, we take an alternative approach by developing iterative algorithms for reconstructing a minimum (or maximum) phase signal from the phase or magnitude of its Fourier transform. Specifically, we develop algorithms which impose causality in the time domain and the given phase or magnitude in the frequency domain, in an iterative fashion.

Iterative algorithms similar to those we discuss here have been useful in a number of areas where partial information in the two domains is available. In particular, the algorithms presented in this paper are similar in style to the Gerchberg-Saxton algorithm [2] and an iterative algorithm by Fienup [3], in alternately incorporating partial information in the time and frequency domains. The Gerchberg-Saxton algorithm recovers a two-dimensional complex signal by iteratively imposing the finite extent of the signal in the space domain and its magnitude in both the space and frequency domains. Similarly, Fienup's algorithm recovers a real two-dimensional signal by iteratively imposing the finite extent and positivity of the signal in the space domain and its magnitude in the frequency domain. Another iteration in this same style recovers a finite length mixed phase signal from the phase of its Fourier transform by imposing a finite length constraint in the time domain and the known phase in the frequency domain [4].

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T. F. Quatieri, Jr. is with Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, MA 02173.

A. V. Oppenheim is with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139.

In this paper, we begin in Section II with a discussion of a number of equivalent conditions for a sequence to be minimum phase. In Sections III and IV, we use these conditions in developing two iterative reconstruction algorithms for minimum phase signals, one for reconstruction when the phase is known and the other for reconstruction when the magnitude is known.

In Section V, we discuss the discrete Fourier transform (DFT) realizations of the algorithms and illustrate the reconstruction process with examples.

In Section VI, we propose the use of the algorithms of Sections III and IV in implementing the Hilbert transform. Of particular importance is reconstruction of the log-magnitude from phase since the proposed iterative approach requires only the principal value of the phase, while the direct DFT implementation of the Hilbert transform requires the unwrapped phase [5]. The proposed technique, therefore, avoids problems typical of phase unwrapping such as detection of the discontinuities in the principal value of the phase [1], [6]. Also, in Section VI, we suggest the use of this new approach to implementing the Hilbert transform as the basis for a phase unwrapping algorithm.

## II. THE MINIMUM PHASE CONDITION

In general, a signal cannot be uniquely specified by only the phase or magnitude of its Fourier transform. However, one condition under which the magnitude and phase are related is the minimum phase condition and under this condition a signal can be uniquely recovered from the magnitude of its Fourier transform or to within a scale factor, from the phase of its Fourier transform. In this section, we discuss a number of equivalent conditions for a signal to be minimum phase. These conditions will be of particular importance in Section III in developing the iterative algorithms.

In the following discussion we restrict the $z$ transform of the sequence $h(n)$ to be a rational function, which we express in the form

$$H(z) = Az^{n_o} \frac{\prod_{k=1}^{M_i} (1 - a_k z^{-1}) \prod_{k=1}^{M_o} (1 - b_k z)}{\prod_{k=1}^{P_i} (1 - c_k z^{-1}) \prod_{k=1}^{P_o} (1 - d_k z)} \tag{1}$$

where $|a_k|$, $|b_k|$, $|c_k|$, and $|d_k|$ are less than or equal to unity, $z^{n_o}$ is a linear phase factor, and $A$ is a scale factor. When, in addition, $h(n)$ is stable, i.e., $\Sigma_n |h(n)| < \infty$, $|c_k|$ and $|d_k|$ are strictly less than one.

A complex function $H(z)$ of a complex variable $z$ is defined to be minimum phase if it and its reciprocal $H^{-1}(z)$ are both analytic for $|z| \geqslant 1$. A minimum phase sequence is then defined as a sequence whose $z$ transform is minimum phase. For $H(z)$ rational, as in (2), the minimum phase condition excludes poles or zeros on or outside the unit circle in the $z$ plane or at infinity. As a consequence, the factors of the form $(1 - b_k z)$ corresponding to zeros on or outside the unit circle and the factors of the form $(1 - d_k z)$ corresponding to poles on or outside the unit circle will not be present. Furthermore, in (1), $n_o = 0$ to exclude poles or zeros at infinity. Thus, for $H(z)$ minimum phase, (1) reduces to

$$H(z) = A \frac{\prod_{k=1}^{M_i} (1 - a_k z^{-1})}{\prod_{k=1}^{P_i} (1 - c_k z^{-1})} \tag{2}$$

where $|a_k|$ and $|c_k|$ are both strictly less than unity.

From (2) other conditions can be formulated for a signal to be minimum phase. Two conditions in particular which we discuss below are particularly useful in the context of the iterative algorithms to be discussed in Sections III and IV.

*Minimum Phase Condition A*

Consider $h(n)$ stable and $H(z)$ rational in the form of (1) with no zeros on the unit circle. A necessary and sufficient condition for $h(n)$ to be minimum phase is that $h(n)$ be causal, i.e., $h(n) = 0, n < 0$, and $n_o$ in (1) be zero.

From (2), it follows that these conditions are necessary. To show that they are sufficient, we want to show that they force (1) to reduce to (2). Clearly, factors of the form $(1 - d_k z)$, $|d_k| < 1$ in the denominator introduce poles outside the unit circle which would violate the causality condition since $h(n)$ is restricted to be stable. With $n_o = 0$ in (1), factors of the form $(1 - b_k z)$ would introduce positive powers of $z$ in the Laurent expansion of $H(z)$, requiring $h(n)$ to have some nonzero values for negative values of $n$, thereby again violating the causality condition. Therefore, these factors cannot be present and with $n_o = 0$, (1) reduces to (2). Finally, because our condition assumes $h(n)$ is stable and that $H(z)$ has no zeros on the unit circle, $h(n)$ is minimum phase.

The above minimum phase conditions require that $h(n)$ be causal and that the unwrapped phase function have no linear phase component. Another slightly different set of necessary and sufficient conditions for a signal to be minimum phase can be stated as follows.

*Minimum Phase Condition B*

Consider $h(n)$ stable and $H(z)$ rational in the form of (1) with no zeros on the unit circle. A necessary and sufficient condition for $h(n)$ to be minimum phase is that $h(n)$ be causal, i.e., $h(n) = 0$, $n < 0$ and $h(0) = A$ where $A$ is the scale factor of (1).

Again, from (2) it follows that these conditions are necessary since (2) has no poles or zeros outside the unit circle or at infinity, guaranteeing causality, and from the initial value theorem $h(0) = \lim_{z \to \infty} H(z) = A$. To demonstrate that these conditions are sufficient, we note that again causality of $h(n)$ will eliminate factors of the form $(1 - d_k z)$ in the denominator of (1). Furthermore, since the conditions require that $h(n)$ be causal, the initial value theorem can be applied with the result that

$$h(0) = \lim_{z \to \infty} H(z) = \lim_{z \to \infty} A z^{n_o} \prod_{k=1}^{M_o} (1 - b_k z). \tag{3}$$

Since $h(0) = A$,

$$\lim_{z \to \infty} z^{n_o} \prod_{k=1}^{M_o} (1 - b_k z) = 1 \tag{4}$$

and since $|b_k| < 1$ this requires that $n_o = 0$ and the $b_k$'s be equal to zero. Thus, again (1) reduces to (2).

Another condition which can be shown to be equivalent to minimum phase condition A or B or our original definition of a minimum phase sequence is that the log-magnitude and unwrapped phase of $H(\omega)$ are related through the Hilbert transform [1]. The Hilbert transform relation guarantees that a minimum phase sequence can be uniquely specified from the Fourier transform magnitude and, to within a scale factor, from the Fourier transform phase.

One technique for minimum phase signal reconstruction from phase or magnitude relies on a DFT implementation of the Hilbert transform [5]. In the next two sections, we take an alternate approach which invokes an iterative computation. Motivated by the minimum phase condition A, when the phase is given we impose, in an iterative fashion, causality in the time domain and the known phase in the frequency domain. When the resulting sequence satisfies minimum phase condition A and has the given phase, it must equal $h(n)$ to within a scale factor. Likewise, motivated by the minimum phase condition B, when the magnitude is given, we impose, in an iterative fashion, causality and the initial value $h(0)$ in the time domain, and the known magnitude in the frequency domain. When the algorithm results in a sequence which satisfies minimum phase condition B and has the given magnitude, it must equal $h(n)$.

## III. An Iterative Algorithm for Signal Reconstruction from Phase

The iterative algorithm for reconstructing a minimum phase signal from its phase function is shown in Fig. 1. The function $h_k(n)$ represents the signal estimate on the $k$th iteration and $\tilde{h}_{k+1}(n) = h_k(n)\, u(n)$ where $u(n)$ is the unit step function. The function $\theta_h(\omega)$ is the known phase and $M_{k+1}(\omega)$ and $\theta_{k+1}(\omega)$ are the Fourier transform magnitude and phase of $\tilde{h}_{k+1}(n)$, respectively.

The algorithm begins with an initial guess $M_0(\omega)$ of the desired Fourier transform magnitude and the inverse Fourier transform of $M_0(\omega) \exp[j\theta_h(\omega)]$ is taken. This step yields $h_0(n)$, the initial estimate of $h(n)$. Next, causality is imposed so that $h_0(n)$ is set to zero for $n < 0$ to obtain $\tilde{h}_1(n)$. The

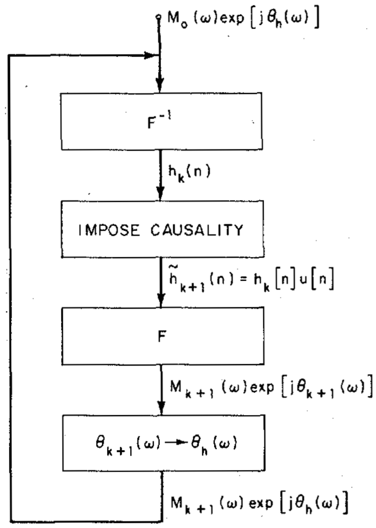


Fig. 1. Iterative algorithm to recover $h(n)$ from its phase.

phase of the Fourier transform of $\tilde{h}_1(n)$ is then replaced by the given phase and the procedure is repeated.

We now show that the mean-square error between $h(n)$ and $h_k(n)$ or, equivalently, between their respective Fourier transforms, $H(\omega) = M_h(\omega)\exp[j\theta_h(\omega)]$ and $H_k(\omega) = M_k(\omega)\exp[j\theta_h(\omega)]$, is nonincreasing on successive iterations. The mean-square error on the $k$th iteration from Parseval's Theorem can be written as

$$\begin{aligned} E_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega) - H_k(\omega)|^2\, d\omega \\ &= \sum_n |h(n) - h_k(n)|^2 \\ &= \sum_{n<0} |h(n) - h_k(n)|^2 + \sum_{n\geq 0} |h(n) - h_k(n)|^2. \end{aligned} \tag{5}$$

Since $\tilde{h}_{k+1}(n) = h_k(n)\,u(n)$

$$|h(n) - h_k(n)| = |h(n) - \tilde{h}_{k+1}(n)|, \qquad n \geq 0 \tag{6}$$

and

$$|h(n) - h_k(n)| \geq |h(n) - \tilde{h}_{k+1}(n)| = 0, \qquad n < 0. \tag{7}$$

Summing (6) and (7) over all $n$, we obtain

$$\begin{aligned} E_k &= \sum_n |h(n) - h_k(n)|^2 \\ &\geq \sum_n |h(n) - \tilde{h}_{k+1}(n)|^2. \end{aligned} \tag{8}$$

Next, from Parseval's Theorem, we write (8) as

$$E_k \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega) - \tilde{H}_{k+1}(\omega)|^2\, d\omega \tag{9}$$

$$\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |M_h(\omega)\exp[j\theta_h(\omega)] - M_{k+1}(\omega)\exp[j\theta_{k+1}(\omega)]|^2\, d\omega.$$

With the triangle inequality for vector differences, we have at each frequency $\omega$:

$$\begin{aligned} &|M_h(\omega)\exp[j\theta_h(\omega)] - M_{k+1}(\omega)\exp[j\theta_{k+1}(\omega)]| \\ &\quad \geq |M_h(\omega)\exp[j\theta_h(\omega)]| - |M_{k+1}(\omega)\exp[j\theta_{k+1}(\omega)]| \\ &\quad \geq |M_h(\omega) - M_{k+1}(\omega)|. \end{aligned} \tag{10}$$

Therefore, from (9) and (10), and the identity $|\exp[j\theta_h(\omega)]|^2 = 1$:

$$\begin{aligned} E_k &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |M_h(\omega) - M_{k+1}(\omega)|^2\, d\omega \\ &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |M_h(\omega)\exp[j\theta_h(\omega)] - M_{k+1}(\omega)\exp[j\theta_h(\omega)]|^2\, d\omega \\ &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega) - H_{k+1}(\omega)|^2\, d\omega \\ &\geq E_{k+1}. \end{aligned} \tag{11}$$

Since $E_k$ is, therefore, nonincreasing and has a lower bound of zero, $E_k$ must converge to a unique limit [7]. The nonincreasing nature of $E_k$, however, is not sufficient to guarantee that the iterates $h_k(n)$ converge. Nevertheless, *if* a converging solution with a rational $z$ transform exists, we can show that

$$\lim_{k\to\infty} h_k(n) = \alpha h(n) \tag{12}$$

where $\alpha$ is a positive constant.

To see this, note from (6), (7), and (10) that the equality in (11) holds if and only if $h_k(n) = \tilde{h}_{k+1}(n) = 0$ for $n < 0$, and $\theta_h(\omega) = \theta_{k+1}(\omega)$. Therefore, since $\theta_h(\omega)$ contains no linear phase component (i.e., $n_o = 0$), if $h_k(n)$ converges to a sequence whose $z$ transform is of the form in (1), the converging solution must satisfy the minimum phase condition A. Consequently, the converging solution is minimum phase with phase $\theta_h(\omega)$, and (12) must hold.[1]

When $h(n)$ is of finite duration (i.e., $H(z)$ has no poles), we can impose not only causality, but also a finite duration constraint within the iteration. Under these particular constraints, the DFT realization of our iterative procedure (see Section V) always converges to a limit of the form in (12) [8].

## IV. An Iterative Algorithm for Signal Reconstruction from Magnitude

In this section we present an iterative algorithm for reconstruction of a minimum phase signal from the magnitude of its Fourier transform. The algorithm is shown in Fig. 2. The function $h_k(n)$ represents the signal estimate on the $k$th iteration and $\tilde{h}_{k+1}(n)$ is defined by

$$\tilde{h}_{k+1}(n) = \begin{cases} h_k(n), & n > 0 \\ h(0), & n = 0 \\ 0, & n < 0. \end{cases} \tag{13}$$

[1] The constant $\alpha$ in (12) is constrained to be positive since a negative value introduces an additive factor of $\pi$ into the phase function.

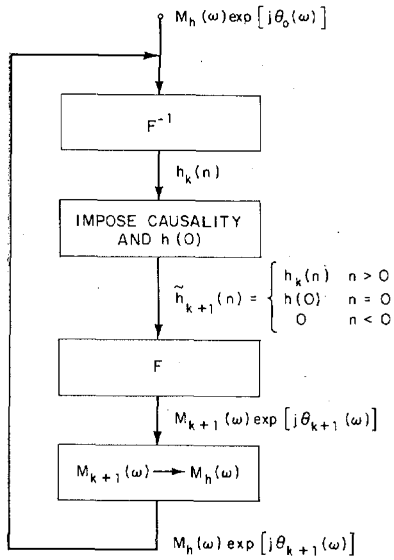

Fig. 2. Iterative algorithm to recover $h(n)$ from its magnitude.

The function $M_h(\omega)$ is the known magnitude and $M_{k+1}(\omega)$ and $\theta_{k+1}(\omega)$ are the Fourier transform magnitude and phase of $\tilde{h}_{k+1}(n)$, respectively.

The algorithm begins with an initial guess $\theta_0(\omega)$ of the desired phase, and the inverse transform of $M_h(\omega) \exp [j\theta_0(\omega)]$ is taken. This step yields $h_0(n)$, the initial estimate of $h(n)$. Next, on the basis of the minimum phase condition B, causality and the known value of $h(0)$ are imposed so that $h_0(n)$ is set to zero for $n < 0$ and set to $h(0)$ for $n = 0$, to obtain $\tilde{h}_1(n)$. The magnitude of the Fourier transform of $\tilde{h}_1(n)$ is then replaced by the given magnitude and the procedure is repeated.

It has not been possible to show that the mean-square error, as considered in Section III, is nonincreasing for this algorithm. However, an error function that is nonincreasing is the mean-square difference between the known magnitude and the estimate $M_k(\omega)$ on each iteration, i.e.,

$$E_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |M_h(\omega) - M_k(\omega)|^2 \, d\omega. \tag{14}$$

To show that $E_k$ is nonincreasing, we first use the identity $|\exp[j\theta_k(\omega)]|^2 = 1$ to express $E_k$ as

$$\begin{aligned} E_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |M_h(\omega) - M_k(\omega)|^2 \, |\exp[j\theta_k(\omega)]|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |M_h(\omega) \exp[j\theta_k(\omega)] \\ &\quad - M_k(\omega) \exp[j\theta_k(\omega)]|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_k(\omega) - \tilde{H}_k(\omega)|^2 \, d\omega. \end{aligned} \tag{15}$$

From Parseval's Theorem, (15) is given in the time domain by

$$E_k = \sum_n |h_k(n) - \tilde{h}_k(n)|^2. \tag{16}$$

From (13), it follows that

$$|h_k(n) - \tilde{h}_k(n)| \geq |h_k(n) - \tilde{h}_{k+1}(n)| = 0, \quad n > 0 \tag{17}$$

and

$$|h_k(n) - \tilde{h}_k(n)| = |h_k(n) - \tilde{h}_{k+1}(n)|, \quad n \leq 0. \tag{18}$$

Summing (17) and (18) over all $n$, we obtain

$$E_k = \sum_n |h_k(n) - \tilde{h}_k(n)|^2 \geq \sum_n |h_k(n) - \tilde{h}_{k+1}(n)|^2. \tag{19}$$

Next, we apply the triangle inequality for vector differences, to yield

$$|H_k(\omega) - \tilde{H}_{k+1}(\omega)| \geq |H_k(\omega)| - |\tilde{H}_{k+1}(\omega)|. \tag{20}$$

Therefore, we have from Parseval's Theorem, and (19) and (20)

$$\begin{aligned} E_k &\geq \sum_n |h_k(n) - \tilde{h}_{k+1}(n)|^2 \\ &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_k(\omega) - \tilde{H}_{k+1}(\omega)|^2 \, d\omega \\ &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |M_h(\omega) - M_{k+1}(\omega)|^2 \, d\omega \\ &\geq E_{k+1}. \end{aligned} \tag{21}$$

Since $E_k$ is nonincreasing and has a lower bound of zero, it must converge to a limit point [7].

As with the algorithm in Section III, although we have shown that the error $E_k$ is nonincreasing, we have not shown that the iterates $h_k(n)$ converge. However, *if* the iterates converge to a sequence whose $z$ transform is rational with no zeros on the unit circle and which is causal with initial value $h(0)$, from the minimum phase condition B, the converging solution must be minimum phase. Consequently, if in addition the magnitude of the Fourier transform of the converging solution equals $M_h(\omega)$, the solution is the unique minimum phase sequence associated with $M_h(\omega)$, i.e., $h(n)$.

The convergence of $h_k(n)$ has yet to be rigorously proven even when a finite length constraint is imposed within the iteration [8]. Empirically, however, we have found the DFT realization of the algorithm to always converge. In the next section, we shall illustrate the convergence of $h_k(n)$ to $h(n)$ with an example.

## V. Realizations of the Iterative Algorithms Using the DFT

Since the iterative algorithms will be implemented on a digital computer, we can compute a Fourier transform at only a finite number of points. In particular, we shall use the DFT.

One consequence of the DFT realization is that our desired sequence $h(n)$ must be of finite duration. Imposing a finite duration constraint within the iterations, however, does not change the nonincreasing nature of the error functions, as can be seen from (8) and (19).

A second consequence of the DFT realization is that only uniformly spaced samples of the phase and magnitude functions are available. Nevertheless, it is again possible to show

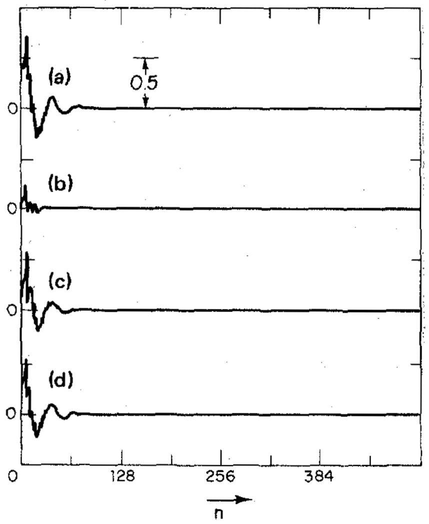


Fig. 3. Convergence of $h_k(n)$ in example 1: (a) original, (b) 1 iteration, (c) 5 iterations, (d) 45 iterations.

that the nonincreasing nature of $E_k$ is not altered when we use samples of the magnitudes and Fourier transforms in the expressions for $E_k$ in (5) and (14) [9], [10].

Finally, questions of convergence need to be addressed. Consider first, minimum phase reconstruction from phase samples. When $H(z)$ is constrained to have no conjugate reciprocal zero pairs and no zeros on the unit circle, a unique sequence $h(n)$ of length $M$ (to within a scale factor) is guaranteed when given $M - 1$ or more phase samples of $\theta_h(\omega)$ in the open frequency interval $(0, \pi)$ [9]. A minimum phase sequence, in particular, satisfies these constraints. Therefore, the DFT realization of the iterative algorithm to reconstruct a minimum phase sequence from its phase samples can be implemented with a DFT of length $N \geqslant 2M$. Furthermore, this iteration will converge to $\alpha h(n)$ for $0 \leqslant n \leqslant N - 1$ where $\alpha$ is positive [8].

Consider next, the dual problem of developing a DFT realization of the iterative algorithm to recover a minimum phase sequence of length $M$ from a magnitude function. In this case, there exists only one $M$ point sequence, i.e., the minimum phase sequence $h(n)$ when $h(0)$ is specified along with $M$ or more uniformly spaced samples of the magnitude in the half-open frequency interval $[0, \pi)$ [10]. Therefore, a DFT realization of the iteration can be implemented with DFT length $N \geqslant 2M - 1$. If the algorithm converges to a causal sequence of length $M$ with initial value $h(0)$ and the known magnitude samples, the converging solution must equal $h(n)$ for $0 \leqslant n \leqslant N - 1$.

To illustrate, we now consider two examples where the DFT length is 512 points, which is twice the length of $h(n)$. In the first example, the initial magnitude guess is unity, and in the second example the initial phase guess is zero.

*Example 1: Signal Reconstruction from Phase*

Consider a 256-point minimum phase signal $h(n)$ illustrated in Fig. 3.[2] The phase is known and we wish to reconstruct

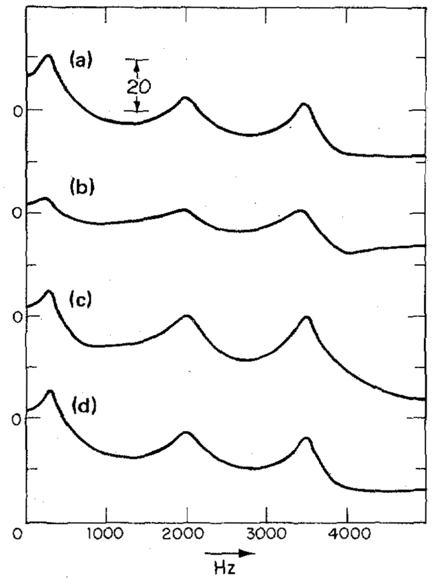


Fig. 4. Convergence of log $|H_k(\omega)|$ (in decibels) in example 1: (a) original, (b) 1 iteration, (c) 5 iterations, (d) 45 iterations.

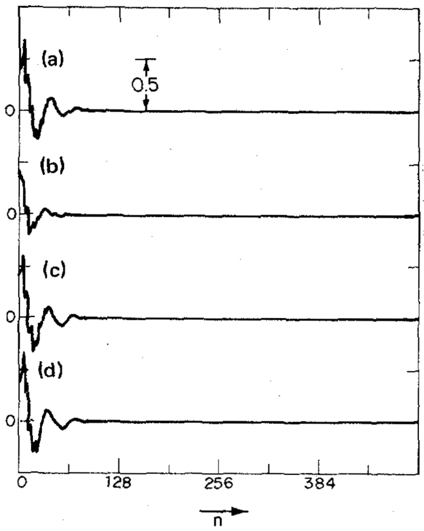


Fig. 5. Convergence of $h_k(n)$ in example 2: (a) original, (b) 1 iteration, (c) 5 iterations, (d) 25 iterations.

$h(n)$. The functions $h_k(n)$ and log $[M_k(\omega)]$ are depicted in Figs. 3 and 4 along with the originals for $k$ equal to 1, 5, and 45. The signal $h_k(n)$ (to within a multiplicative constant) and the spectrum log $[M_k(\omega)]$ (to within an additive constant) are indistinguishable from the originals after 45 iterations.

*Example 2: Signal Reconstruction from Magnitude*

In this example, we consider the sequence of example 1, but where the Fourier transform magnitude is given. The functions $h_k(n)$ and $\theta_k(\omega)$ are depicted in Figs. 5 and 6 with the originals for $k$ equal to 1, 5, and 25. The functions $h_k(n)$ and $\theta_k(\omega)$ are indistinguishable from the originals after 25 iterations.

## VI. A Basis for Implementation of the Hilbert Transform and Phase Unwrapping

In this section, we propose two computational algorithms based on the procedures of Sections III and IV: 1) an iterative

[2]The $z$ transform of this signal consists of two complex pole pairs and one complex zero pair all within the unit circle. For $n > 256$, $h(n)$ has decayed to effectively zero.

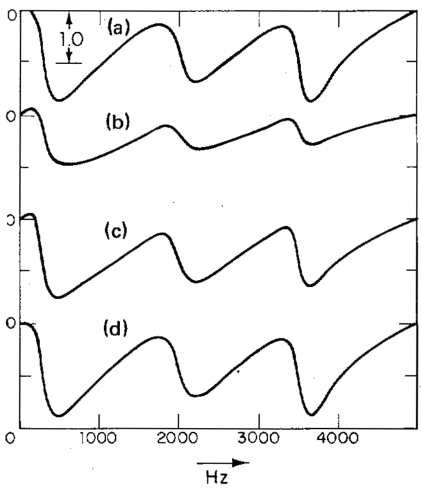


Fig. 6. Convergence of $\theta_k(\omega)$ (in radians) in example 2: (a) original, (b) 1 iteration, (c) 5 iterations, (d) 25 iterations.

approach to computing the Hilbert transform, and 2) the potential use of 1) as the basis of a phase unwrapping algorithm.

For a minimum phase signal, the log-magnitude and phase of the Fourier transform are related through the Hilbert transform and the direct implementation of the Hilbert transform using the DFT has been extensively investigated [5]. One disadvantage of this implementation is that in computing the log-magnitude from the phase, samples of the unwrapped phase are required and are often difficult to compute.

An alternative to the direct implementation of the Hilbert transform exploits the iterative algorithms of Sections III and IV. When the phase is given, through the use of the algorithm in Section III, $\alpha h(n)$ is first obtained from the phase and, in particular, does not require samples of the unwrapped phase. From $\alpha h(n)$ the log-magnitude of $\alpha H(\omega)$, representing the Hilbert transform of the phase to within an additive factor is then computed. Furthermore, with a fixed DFT length, by increasing the number of iterations we can come arbitrarily close to samples of the log-magnitude. The direct approach, on the other hand, requires an increase in the DFT length for an increase in accuracy [1], [6].

A similar procedure can, of course, be applied through the use of the iterative algorithm in Section IV to implement the Hilbert transform of a given log-magnitude function. If $h(0)$ is not known *a priori* [recall (13)], it can be obtained (at least in theory) from the magnitude, although in practice $h(0)$ can be computed only approximately [1]. However, it was found empirically that the iterates always converge to $h(n)$ when only causality is imposed in the time domain (i.e., $h(0)$ is assumed unknown) and the initial phase $\theta_0(\omega)$ is set to zero.

This indirect approach to computing the Hilbert transform suggests a potential alternative to available phase unwrapping algorithms [1], [6]. Let $\theta(\omega)$ denote the desired unwrapped phase of the Fourier transform $H(\omega)$ and $\theta_p(\omega)$ its value modulo $2\pi$. Furthermore, assume that the linear phase component of $H(\omega)$, i.e., $n_o$ in (1), is known. The proposed phase unwrapping algorithm proceeds as follows.

1) Remove the linear phase component to obtain the principal value of the phase of $H(\omega) \exp[-jn_o\omega]$, denoted by $\hat{\theta}_p(\omega)$.

2) Apply the iterative algorithm of Section III with a causality constraint and with phase $\hat{\theta}_p(\omega)$ to obtain a minimum phase sequence $h_{mp}(n)$.

3) Compute $\log |H_{mp}(\omega)|$ where $H_{mp}(\omega)$ is the Fourier transform of $h_{mp}(n)$.

4) Apply the Hilbert transform to $\log |H_{mp}(\omega)|$ to obtain the unwrapped phase function $\theta(\omega) - n_o\omega$.

5) Add the linear phase component to obtain the desired unwrapped phase.

Of particular interest is step 2 which yields the same minimum phase sequence $h_{mp}(n)$ that would be obtained by a Hilbert transform of the unwrapped phase, but bypasses the need for phase unwrapping. This algorithm has performed successfully on a number of simple mixed-phase sequences (i.e., two and three poles and/or zeros) when the linear phase component was known exactly. Furthermore, it yielded the correct phase function when poles and zeros were placed close to the unit circle.

There are several potential difficulties in the use of this algorithm. First, the minimum phase sequence $h_{mp}(n)$ derived from the iteration is of infinite extent regardless of whether the original sequence $h(n)$ is of finite duration [10]. Therefore, a possible problem with aliasing arises. The DFT length must be sufficiently large so that the minimum phase sequence $h_{mp}(n)$ decays effectively to zero. In particular, when $h_{mp}(n) = 0$ for $n > M$, the DFT length, from our discussion in Section V, should be at least $2M$.

One possible procedure for removing the effects of aliasing invokes the principal value of the phase $\theta_p(\omega)$ in a style similar to that in [6]. In particular, we might consider adding a step 6 of the following form.[3]

6) At each frequency, subtract $2\pi$ from the unwrapped phase estimate until the result is between $-\pi$ and $\pi$. Then add this multiple of $2\pi$ so found to the principal value of the original phase function $\theta_p(\omega)$. This will give an unwrapped phase free of the aliasing errors introduced in the iterative process.

A second potential difficulty is the requirement that the linear phase factor of $H(z)$ be known. Often, *a priori* knowledge of such a factor is difficult to obtain. One means of obtaining a linear phase estimate is to numerically integrate the phase derivative [1]. The sensitivity[4] of our proposed phase unwrapping algorithm to deviations from the true linear phase in such an estimate is an area which needs to be explored if a practical algorithm is to evolve.

## VII. Summary and Conclusions

In this paper, we have developed iterative algorithms for reconstructing a minimum phase sequence from either the phase or magnitude of its Fourier transform. When the phase is known, the mean-square error between the desired Fourier transform and its estimate was shown to be nonincreasing on successive iterations. Likewise, when the magnitude is given,

[3]The authors acknowledge with thanks this suggestion by Dr. R. W. Schafer.

[4]When the residual linear phase is negative (so that $n_o$ in $z^{n_o}$ is negative), a causal converging solution of the form in (1) derived from step 2 of our proposed algorithm will be mixed phase. In this case, the number of zeros outside the unit circle cannot be greater than $|n_o|$. When $n_o$ is positive, no causal converging solution of the form in (1) can exist.

on successive iterations the mean-square error between the known magnitude and its estimate is nonincreasing. In addition, we noted that convergence of the iteration with known phase samples (i.e., the DFT realization) has been demonstrated, but convergence of the iteration with magnitude samples has been observed only empirically.

Finally, we suggested two computational algorithms based on the iterative procedures: 1) a new means of implementing the Hilbert transform which avoids the need of an unwrapped phase, and 2) a new procedure for phase unwrapping.

The iterative algorithms, as presented, rely on exact knowledge of the magnitude, phase, and the initial value of the desired signal. Sensitivity to the inexactness of these quantities, to quantization noise, and other forms of degradation is not understood and is an important area for future research.

In practice, we have found that the iterative algorithms converge sometimes slowly (e.g., after several hundred iterations) and sometimes quickly (e.g., after a few iterations). Consequently, determining rates of convergence in terms of characteristics of the minimum phase signal and initial magnitude or phase estimates, and methods of speeding up convergence need to be explored.

Another area being considered is the interchange of the signal reconstruction problems. In particular, we have found empirically that when $M_h(\omega)$ and $\theta_h(\omega)$ are interchanged through $j \log H(\omega)$, a slightly modified version of the iterative algorithm of Section III, requiring a phase function, will recover $h(n)$ from its magnitude. Likewise, when the phase is known, $h(n)$ is recovered by a procedure similar to the iteration in Section IV which requires a magnitude function. These results have led to some interesting theoretical speculations about the duality of the reconstruction problems and their iterative solutions.

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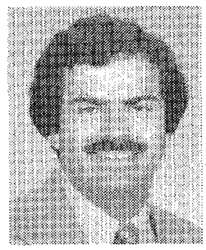
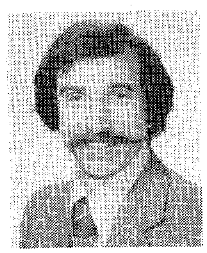


**Thomas F. Quatieri, Jr.** (S'70-S'78-M'79) was born in Somerville, MA, on January 31, 1952. He received the B.S. degree (summa cum laude) from Tufts University, Medford, MA in 1973, and the S.M., E.E., and Sc.D. degrees from the Massachusetts Institute of Technology (M.I.T.), Cambridge in 1975, 1977, and 1979, respectively.

From 1973 to 1975, he was a Teaching Assistant and from 1975 to 1979 he was a Research Assistant in the area of digital signal processing, both within the Department of Electrical Engineering, M.I.T. His research for the Master's degree involved the design of two-dimensional digital filters and for the Sc.D. it involved phase estimation with application to speech analysis-synthesis. He is presently a Research Staff Member at M.I.T., Lincoln Laboratory, where he is working on problems in multidimensional digital signal processing with applications to image processing.

Dr. Quatieri is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.



**Alan V. Oppenheim** (S'57-M'65-SM'71-F'77) received the S.B. and S.M. degrees in 1961 and the Sc.D. degree in 1964, all in electrical engineering, from the Massachusetts Institute of Technology (M.I.T.), Cambridge.

From 1961 to 1964 he was a member of the M.I.T. Research Laboratory of Electronics and an Instructor in the Department of Electrical Engineering. During this period his research activities involved the application of modern algebra to the characterization of nonlinear systems and the development of homomorphic filtering as an approach to some nonlinear filtering problems. In 1964 he joined the faculty in the Department of Electrical Engineering and Computer Science and became a Staff Member of the M.I.T. Research Laboratory of Electronics. From 1967 to 1969, he was a staff member of the M.I.T. Lincoln Laboratory, working in the areas of digital signal processing and speech communication. In 1969, he was appointed as an Associate Professor and, in 1976, was appointed as a full Professor, both in the Department of Electrical Engineering and Computer Science at M.I.T. During the academic year 1972-1973, on sabbatical leave, he was a Guggenheim Fellow at the University of Grenoble, Grenoble, France. From 1976 to 1978, he held the Cecil H. Green Distinguished Chair in Electrical Engineering and Computer Science. From 1978 to 1980 he was Associate Head of the Data Systems Division at M.I.T., Lincoln Laboratory. He is currently Professor of Electrical Engineering and Computer Science at M.I.T. His research interests are in the area of digital signal processing. He is coauthor of a text on *Digital Signal Processing* and Editor of a book on *Applications of Digital Signal Processing.* He is also coauthor of a text on *Signals and Systems* to be published.

Dr. Oppenheim is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi. In 1969 he received the IEEE G-AE Senior Award for "authoring a paper of exceptional merit," and in 1976 he received the IEEE Acoustics, Speech, and Signal Processing Society Technical Achievement Award "in recognition of fundamental contributions to the development of homomorphic processing techniques for speech and other signals." In 1980 he received the IEEE Acoustics, Speech, and Signal Processing Society Award "in recognition of sustained leadership in the field of digital signal processing by your innovative research, writing of pioneering textbooks, and inspiring teaching."