Efficient Rational Sampling Rate Alteration Using IIR Filters

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***Abstract*—The problem of changing the sampling rate of a signal by a rational factor of L/M is discussed. It is shown that infinite impulse response (IIR) filters can be efficiently implemented using the polyphase decomposition. The computational cost of the recursive and nonrecursive parts of the interpolation filter are considered separately, and a gain in efficiency of a factor of $LM/(L+M-1)$ is achieved for the recursive part. This gain is only significant when both L and M are larger than one.**

***Index Terms*—IIR filters, multirate signal processing, polyphase decomposition, rational sampling rate converter.**

I. Introduction

SAMPLING rate conversion is the process of taking a sequence, $x[n]$, which is associated with a sampling rate, f_x, and converting it to another sequence, $y[n]$, which is associated with a different sampling rate, f_y. In other words, if we have $x[n]$ which could have been obtained by sampling a particular bandlimited continuous-time signal at a rate of f_x, then $y[n]$ could also have been obtained by sampling the same signal at a rate of f_y. This letter deals specifically with rational sampling rate conversion, so the conversion ratio $f_y/f_x = L/M$, where L and M are relatively prime integers.

A. Notation and Definitions

The notation and terminology used in this letter follows Vaidyanathan [1]. We utilize only type-1 polyphase decomposition, so all polyphase components referred to should be assumed to be of type-1. Fig. 1 summarizes the identities for multirate systems which are used here.

B. Problem Statement

In its simplest form, a rational sampling rate converter can be represented by the system in Fig. 2, where $H(z)$ is the system function of a low-pass filter, with a cut off frequency $\omega_c = \min\{\pi/L, \pi/M\}$.

We wish to find $H(z)$ such that its frequency response approximates an ideal lowpass filter to within some given specification, and which requires the least amount of computation when implemented.

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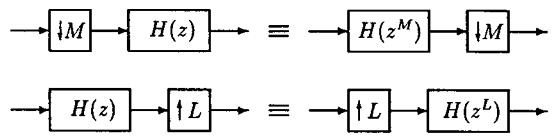

Fig. 1. Identities for multirate systems.

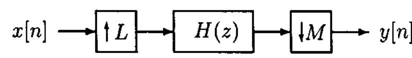

Fig. 2. Simplistic system for rational sampling rate conversion.

C. Background

Generally, $H(z)$ is chosen to be finite impulse response (FIR) because it is known that FIR filters can be implemented very efficiently using the polyphase decomposition, as described by Vaidyanathan [1]. Bellanger *et al.* [2], [3] as well as Crochiere and Rabiner [4] have discussed the use of infinite impulse response (IIR) filters for sampling rate conversion, but they did not explicitly consider the case of rational noninteger conversion. We will show that this is the only case in which there is a gain in efficiency over the direct implementation for the recursive part.

II. The Efficient Implementation

Let us consider an IIR filter with rational transfer function, $H(z)$. In the most general sense, $H(z)$ is given by

$$H(z) = \frac{\sum_{k=0}^{N_Z} a_k z^{-k}}{1 + \sum_{k=1}^{N_P} b_k z^{-k}} = \frac{a_0 \prod_{k=1}^{N_Z} (1 - \alpha_k z^{-1})}{\prod_{k=1}^{N_P} (1 - \beta_k z^{-1})} \tag{1}$$

and has N_Z zeros and N_P poles. The number of multiplies needed to calculate one sample of $y[n]$ (*multiplies per output sample*, or MPOS) using the direct implementation is $M(N_Z + 1) + MN_P$.

We now use the substitution

$$1 - \beta_i z^{-1} \equiv \frac{1 - \beta_i{}^D z^{-D}}{\sum_{k=0}^{D-1} \beta_i{}^k z^{-k}} \tag{2}$$

for each of the N_P poles in $H(z)$, with $D = L$ for N_L of the poles, and $D = M$ for N_M of the poles, where $N_L + N_M =$

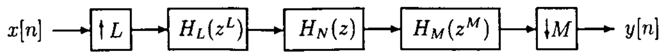

Fig. 3. $H(z)$ replaced by a cascade of three filters that operate at different rates. This system is equivalent to the system shown in Fig. 2.

N_P. This will increase the numerator order by $N_L(L-1)+N_M(M-1)$, so $H(z)$ is now

$$H(z)=\frac{\sum_{k=0}^{N_N} c_k z^{-k}}{\prod_{k=1}^{N_L}(1-\gamma_k{}^L z^{-L})\prod_{k=1}^{N_M}(1-\xi_k{}^M z^{-M})} \tag{3}$$

where $N_N=N_Z+N_L(L-1)+N_M(M-1)$, and each β now becomes a γ or a ξ. This can be viewed as the cascade of three filters:

1) $H_N(z)$, which is the numerator part and thus FIR;
2) $H_L(z^L)$, which includes the part of the denominator that is only a function of z^L;
3) $H_M(z^M)$, which includes the part of the denominator that is only a function of z^M.

This cascade is shown in Fig. 3.

The identities from Fig. 1 can then be used to commute $H_L(z^L)$ with the interpolator and to commute $H_M(z^M)$ with the decimator. $H_N(z)$ can then be replaced by its LM-component polyphase form, so that $E_k(z)$ is the kth polyphase component of $H_N(z)$, i.e., $E_k(z)$ is the z-transform of $e_k[n]=h_N[LMn+k]$. Since $H_N(z)=\sum_{k=0}^{LM-1} z^{-k}E_k(z^{LM})$, we can replace $H_N(z)$ with a parallel structure where each branch is a term in the sum. Then, for the kth term, z^{-k} can be replaced by $z^{-Lka}z^{Mkb}$, where $La-Mb=1$, and a and b are positive integers. Such an a and b always exist since L and M are relatively prime. Applying the identities and exploiting the commutivity of relatively prime interpolators and decimators [1, Sec. 4.2], gives the system shown in Fig. 4.

When $H(z)$ is implemented using this structure, the number of MPOS required is $N_M+(1/L)(N_N+1)+(M/L)N_L=(1/L)(N_Z+1)+((L+M-1)/L)N_P$. This gives a savings of a factor of LM for the numerator and a factor of $LM/(L+M-1)$ for the denominator. The gain in efficiency is the same regardless of how many poles of $H(z)$ are assigned to $H_L(z)$ and how many to $H_M(z)$. However, complex-conjugate pole pairs should not be separated since this would make the filter coefficients complex.

III. Extension to Previous Work

The substitution (2) was known by Bellanger *et al.* [2]. Also, the efficient implementation for the FIR part, $H_N(z)$, was known by Vaidyanathan [1], and in a different form, by Bellanger *et al.* [2], [3], and Crochiere and Rabiner [4]. However, in [2]–[4] the IIR case was never explicitly analyzed for rational conversion. Only integer interpolation, $M=1$, and integer decimation, $L=1$, were analyzed. This means that $LM/(L+M-1)=1$, and so there is no computational savings for the recursive part. Also, in [2] and [3] it is assumed that $N_Z=N_P$. This leads to the misleading conclusion that the computation cannot be reduced beyond half of that for the direct implementation (see [2, p. 113] and [3, p. 281]). The value of a half comes from the fact that the computation for the recursive part stays the same, while the computation for the nonrecursive part is drastically reduced. However, we have shown that when both L and M are larger than one, then there is a gain in efficiency for both the recursive and nonrecursive parts.

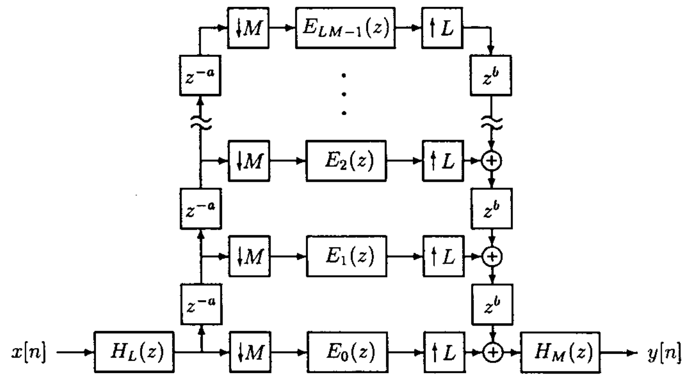

Fig. 4. Structure used for the efficient implementation of $H(z)$. This system is equivalent to the system shown in Fig. 3.

IV. Conclusion

It was shown that IIR filters that can be designed by well-known methods can be efficiently implemented for the case of a rational sampling rate converter. There is a gain in efficiency over the direct implementation by a factor of LM for the numerator and a factor of $LM/(L+M-1)$ for the denominator, where L/M is the conversion ratio. This approach is suboptimal, but is useful in practice because the design of the filter is relatively easy compared to methods for designing optimal multirate filters.

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