

SIGNAL APPROXIMATION USING THE BILINEAR TRANSFORM

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ABSTRACT

This paper explores the approximation properties of a unique basis expansion, which realizes a bilinear frequency warping between a continuous-time signal and its discrete-time representation. We investigate the role that certain parameters and signal characteristics have on these approximations, and we extend the analysis to a windowed representation, which increases the overall time resolution. Approximations derived from the bilinear representation and from Nyquist sampling are compared in the context of a binary detection problem. Simulation results indicate that, for many types of signals, the bilinear approximations achieve a better detection performance.

Index Terms— Signal Representations, Approximation Methods, Bilinear Transformations, Signal Detection

1. INTRODUCTION

Although Nyquist sampling is commonly used in practice, there remain several drawbacks to this representation. For example, the continuous-time (CT) signal must be appropriately bandlimited in order to avoid frequency aliasing distortions. Additionally, if the number of time samples used in a particular computation is constrained, the Nyquist approximation may do a poor job of representing the original signal. For these reasons, it is useful to consider alternative signal representations.

In the 1960's a basis expansion was proposed [1, 2], implementing a nonlinear frequency warping between a CT signal and its discrete-time (DT) representation according to the bilinear transform. Since there is a one-to-one relationship between the two frequency domains, this bilinear expansion theoretically avoids both the bandlimited requirement and the frequency aliasing distortions associated with Nyquist sampling. Furthermore, the DT expansion coefficients can be obtained using a cascade of first-order analog systems. Modern-day integrated circuit technology has made it practical to compute these coefficients through conventional circuit design techniques. Consequently, the bilinear expansion can be considered as an alternative to Nyquist representations in various applications.

In this paper, we explore the approximation performance of the bilinear expansion presented in [1] by drawing from properties of the corresponding basis functions. We consider how various signal characteristics, such as a rational Laplace transform and the energy distribution over time, affect the approximations. Additionally, we examine a modified version of the bilinear representation, in which the CT signal is segmented using a short-duration window. This segmentation procedure helps to improve the overall approximation quality by exploiting properties of the representation.

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The analysis presented in this paper has application in contexts where only a fixed number of DT values can be used to represent a CT signal. For example, in a binary detection problem, one might want to limit the number of digital multipliers used to compute the inner product of two CT signals. Numerical simulations of this scenario suggest that the bilinear expansion achieves a better detection performance than Nyquist sampling for certain signal classes.

2. THE BILINEAR REPRESENTATION

As derived in [1], the network shown in Fig. 1 realizes a one-to-one frequency warping between the Laplace and Z -transform domains according to the bilinear transform. Specifically, the Laplace transform $F(s)$ of the signal $f(t)$ and the Z -transform $F(z)$ of the sequence $f[n]$ are related through the change of variables

$$z = \frac{a + s}{a - s} \quad (1)$$

where a is the real valued parameter indicated in the cascades of Fig. 1. The relationship in (1) maps the entire range of CT frequencies ($j\omega$ -axis) onto the range of unique DT frequencies (unit circle).

Fig. 1 can also be viewed as a basis expansion of the CT signal $f(t)$ in which the Laplace transforms of the basis functions are

$$\Lambda_n(s) = \frac{\sqrt{2a}}{a + s} \left(\frac{a - s}{a + s} \right)^{n-1}, \quad n \geq 1 \quad (2)$$

As given in [1], the corresponding time-domain expressions are

$$\lambda_n(t) = \sqrt{2a}(-1)^{n-1} e^{-at} L_{n-1}(2at)u(t), \quad n \geq 1 \quad (3)$$

where $L_n(\cdot)$ represents a zero-order Laguerre polynomial. It is shown in [1] that $\{\lambda_n(t)\}_{n=1}^{\infty}$ is an orthonormal set of functions which span the space of causal, finite-energy signals. Fig. 2 depicts the bilinear basis functions as the index n and the parameter a vary.

By applying properties of Laguerre polynomials [3, 4] and by defining $\xi = 2n - 3$, the bilinear basis functions can be bounded according to the expression

$$|\lambda_n(t)| \leq C \begin{cases} 1, & 0 \leq t \leq \frac{1}{4a\xi} \\ (4a\xi t)^{-1/4}, & \frac{1}{4a\xi} \leq t \leq \frac{\xi}{2a} \\ \left(2\xi \left[(2\xi)^{1/3} + |2\xi - 2at| \right] \right)^{-1/4}, & \frac{\xi}{2a} \leq t \leq \frac{3\xi}{2a} \\ e^{-\beta t}, & t \geq \frac{3\xi}{2a} \end{cases} \quad (4)$$

where β is a positive constant and $a > 0$. Eq. (4) will be useful when analyzing the approximation properties of this representation.

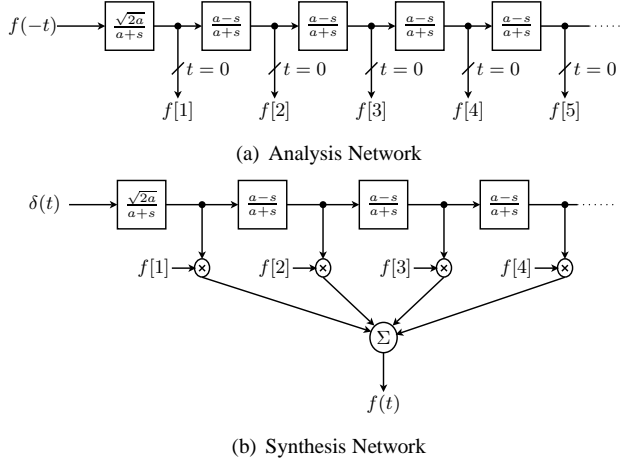


Fig. 1. First order cascade derived from [1] and [2]. The analysis network converts the CT signal $f(t)$ into its DT representation $f[n]$. The synthesis network reconstructs a CT signal from its expansion.

3. BILINEAR APPROXIMATION

In general, representing a CT signal through a basis expansion requires an infinite number of (non-zero) expansion coefficients. We address this issue by retaining an appropriate subset I_M of bilinear expansion terms to form an M -term signal approximation. The approximation error $\epsilon[M]$ is given by

$$\epsilon[M] = \int_0^\infty \left(f(t) - \sum_{n \in I_M} f[n] \lambda_n(t) \right)^2 dt \quad (5)$$

We focus on a nonlinear approximation technique, in which the set I_M is chosen based on the characteristics of $f(t)$. Specifically, because the bilinear basis functions are orthonormal, we retain the largest-magnitude DT coefficients to minimize $\epsilon[M]$. We qualitatively compare approximation performances through the decay of expansion coefficients when sorted by absolute value. As proposed for wavelet approximations in [5, 6], a faster decay corresponds to a smaller M -term approximation error.

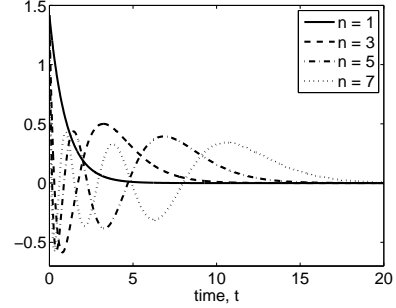
3.1. Effect of the Parameter a

As seen in Fig. 2(b), the behavior of the basis functions is affected by the parameter a . Predictably, this has a direct impact on the approximation performance. As an example, we analyze the bilinear approximations of the windowed sinusoid $f(t) \propto \sin(10t)$ for $0 \leq t < 1$, and then generalize from this information. The signal $f(t)$ has been normalized for unit energy.

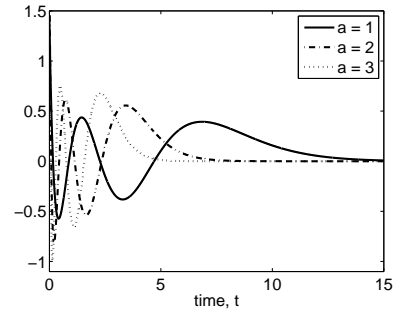
Fig. 3(a)-(d) show the bilinear coefficients $f[n]$ for $a = 1, 10, 100$ and 1000, respectively. As seen, the fastest coefficient decay occurs when $a = 10$, which is the carrier frequency of $f(t)$. An intuitive basis for this result follows by considering the group delay of the all-pass filters in the bilinear first-order cascades.

$$\tau_g(\omega) = \frac{d}{d\omega} \left[\angle \left(\frac{a - j\omega}{a + j\omega} \right) \right] = \frac{2a}{a^2 + \omega^2} \quad (6)$$

For a signal whose energy is tightly concentrated around a frequency of ω_o , the effect of an all-pass filter can be roughly approximated as a time delay of $\tau_g(\omega_o)$. If the signal has approximately



(a) $\lambda_n(t)$ for different index values; $a = 1$



(b) $\lambda_n(t)$ for different values of a ; $n = 5$

Fig. 2. Dependence of the bilinear basis functions on the index n and the parameter a .

finite duration, successive delays of $\tau_g(\omega_o)$ will eventually shift the bulk of its energy beyond the sampling time $t = 0$ in Fig. 1. Once this occurs, the remaining expansion coefficients become very small. Therefore, in order to minimize the number of significant DT coefficients for a narrow-band signal, we should maximize $\tau_g(\omega_o)$ in (6). This is accomplished by setting $a = \omega_o$.

Although many CT signals of interest are not narrow-band, the above analysis suggests the following ‘maximin’ strategy to initialize the value of a : For a signal with most of its information content effectively band-limited to $|\omega| \leq \omega_M$, choose $a = \omega_M$.

Because the group delay in Equation (6) is monotonically decreasing in ω , this strategy guarantees a group delay greater than or equal to $\tau_g(\omega_M)$ for all frequencies in the band $[-\omega_M, \omega_M]$. Unlike a Nyquist anti-aliasing filter, this algorithm does not eliminate high-frequency information. Rather, it tries to capture as much signal energy as possible in the earlier DT coefficients

3.2. Signal Characteristics which Impact Approximation

It is straightforward to verify that signals which have rational Laplace transforms with all poles located at $s = -a$ can be represented exactly using a finite number of bilinear coefficients. In the time domain, this corresponds to polynomials in t weighted by the exponential decay e^{-at} . The approximation performance worsens as the pole location(s) move farther away from $s = -a$. This is confirmed in simulation by examining the sorted coefficient decay of signals $f_k(t) \propto t^3 e^{-kt}$ and $f_p(t) \propto e^{-at} \sin(pt)$ for different values of k and p , respectively.

For signals which do not possess rational Laplace transforms, one important characteristic affecting the bilinear approximation performance is the energy distribution over time. We observe this relation-

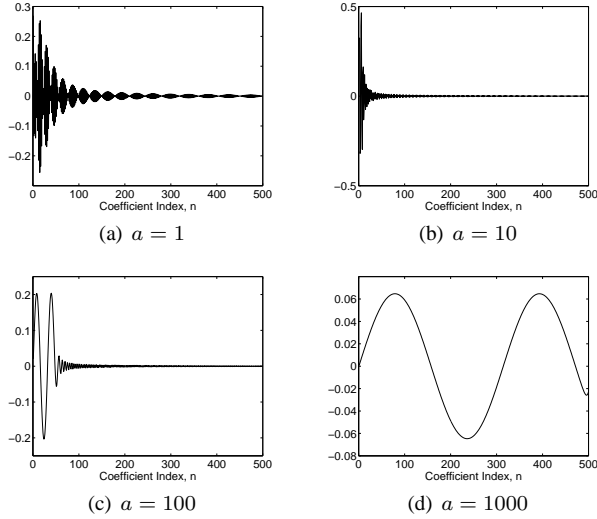


Fig. 3. Bilinear Expansion Coefficients for $f(t) \propto \sin(10t)$

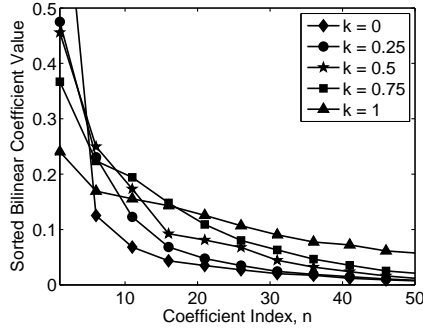


Fig. 4. Sorted DT coefficients of $f_k(t) \propto \text{sinc}(10(t-k))$; $a = 10$

ship by considering the set of signals

$$f_k(t) \propto \begin{cases} \text{sinc}(10(t-k)), & 0 \leq t < 1 \\ 0, & \text{o.w.} \end{cases}$$

where $\text{sinc}(x) = \sin(\pi x)/(\pi x)$ and $f_k(t)$ has been normalized for unit energy. A windowed sinc pulse is chosen because of the large energy concentration around its main lobe in time.

The sorted bilinear coefficients are depicted in Fig. 4. Clearly, as the energy concentration moves farther from the time origin, the approximation performance worsens. This empirical observation is consistent with the basis function behavior as the index increases. Specifically, from Fig. 2 and as suggested by the bound in (4), the time dispersion of $\lambda_n(t)$ increases with n . Consequently, it would be expected that representing signals with primary energy concentration later in time requires basis functions with higher index values.

4. THE WINDOWED REPRESENTATION

As shown, the bilinear expansion is well-suited to signals with energy concentrated early in time. We exploit this property through a windowed representation. Not only does windowing a signal provide greater time resolution, but it can better align signal energy with the time origin of individual segments to improve the approximation.

| # DT Terms | Duration, T | Approximation Type | | |
|------------|---------------|--------------------|------------|------------|
| | | <i>Lin</i> | <i>NLI</i> | <i>NL2</i> |
| 50 | Original | 0.6840 | 0.4567 | — |
| | 0.2 sec | 0.6832 | 0.4532 | 0.0323 |
| | 0.25 sec | 0.5088 | 0.4156 | 0.3044 |
| | 0.34 sec | 0.6533 | 0.4260 | 0.0930 |
| | 0.5 sec | 0.5056 | 0.4062 | 0.3819 |
| 100 | Original | 0.4487 | 0.2166 | — |
| | 0.2 sec | 0.4487 | 0.2319 | 0.0078 |
| | 0.25 sec | 0.4274 | 0.3474 | 0.1822 |
| | 0.34 sec | 0.4234 | 0.1871 | 0.0102 |
| | 0.5 sec | 0.4243 | 0.3412 | 0.2812 |
| 200 | Original | 0.1588 | 0.0231 | — |
| | 0.2 sec | 0.1089 | 0.0422 | 0.0031 |
| | 0.25 sec | 0.3559 | 0.2478 | 0.1014 |
| | 0.34 sec | 0.1032 | 0.0322 | 0.0028 |
| | 0.5 sec | 0.3568 | 0.2455 | 0.1663 |

Table 1. $\epsilon[M]$ for $f(t) \propto \text{sinc}(100(t-0.5))$. A value of $a = 100$ is used for all bilinear expansions.

Mathematically, we treat the original CT signal as a sum of segments $f_k(t)$ created by a finite-duration window $w(t)$ as follows:

$$f(t) = \sum_{k=0}^{\infty} \underbrace{f(t)w(t-kT)}_{f_k(t-kT)} \quad \text{s.t.} \quad \sum_k w(t-kT) = 1, \quad \forall t \quad (7)$$

The choice of window is heavily dependent on the application. For the binary detection problem in Section 5, we segment using a non-overlapping rectangular window. Given our assumption of additive white noise, this window choice simplifies the resulting analysis. The rectangular window may also be advantageous for bilinear approximation, since it yields the shortest segment duration for a given value of T in (7). This may translate to fewer significant expansion coefficients and a smaller M -term approximation error.

We illustrate the benefit of segmentation by approximating the signal

$$f(t) \propto \begin{cases} \text{sinc}(100(t-0.5)), & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The following terminology is used to denote the three approximation methods employed in this work:

Lin The first M coefficients are retained from each segment. Given the cascades in Fig. 1(a), this linear approximation scheme minimizes the system hardware requirements.

NLI The M largest coefficients are retained from each segment.

NL2 The largest coefficient is selected from each segment. For the remaining terms, the largest coefficients overall are selected.

Table 1 shows the approximation errors for different rectangular window sizes and each of the three approximation techniques. There are two main points to note from this data.

First, the segmented representation can achieve a much lower error than the original bilinear representation when using the *NL2* approximation technique, especially as the total number of DT coefficients decreases. This is due to the increased time resolution.

Second, the approximation performance for window durations of $T = 0.25, 0.5$ is notably poorer than for $T = 0.2, 0.34$. This is because the former values of T divide the main lobe in half, causing one segment to possess a rapidly-increasing signal with a large amount of energy. Results from Section 3 suggest that a large number of expansion terms will be required to represent this segment.

5. BINARY DETECTION PROBLEM

In this section we evaluate the bilinear approximation performance in a classical binary detection problem. The goal is to determine whether a desired signal $s(t)$ is present in noise based on the received signal $x(t)$. The channel noise $\eta(t)$ is additive white Gaussian noise (AWGN) with power spectrum, $P_\eta(j\omega) = \sigma_\eta^2$.

The well-known matched filter solution involves comparing the integral of the desired and received signals with a threshold γ

$$\int_0^\infty x(t)s(t)dt \geq \gamma \quad (8)$$

We consider a scenario in which the desired signal is very complex, meaning that we cannot design the analog matched filter $h(t) = s(-t)$. Therefore, (8) must be implemented using DT representations $x[n]$ and $s[n]$. Furthermore, we restrict the number of DT multiplies used to approximate the above integral. For an orthogonal expansion, the detection performance when using M multiplies is inversely-related to the approximation error $\epsilon[M]$.

The detection performances of the bilinear and Nyquist approximations are compared in a series of MATLAB simulations. We consider the following desired signals (normalized to have unit energy):

$$\begin{aligned} s_1(t) &\propto t^2 e^{-150t} u(t) \\ s_2(t) &\propto \begin{cases} \sin(100t), & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \\ s_3(t) &\propto \begin{cases} \text{sinc}(100(t - 0.5)), & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$s_1(t)$ has a rational Laplace transform. $s_2(t)$ is narrow-band in frequency with a constant energy distribution over time. $s_3(t)$ has a wider frequency range with energy concentrated around the main lobe. All signals are sampled approximately 100X faster than the Nyquist rate of the corresponding $s(t)$.

The reduced sets of Nyquist coefficients are obtained by selecting those time samples corresponding to the largest magnitudes in $s(nT_s)$. This is denoted ‘Select Largest Samples’ or *SLS*.

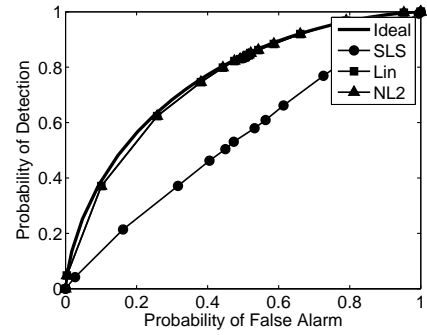
For the bilinear representation, the sequences are segmented with a non-overlapping rectangular window, and the expansion coefficients are computed according to Fig.1(a). The trapezoidal rule for integration is used when numerically simulating the analog systems. To reduce the total number of DT coefficients, we employ the *Lin* and *NL2* techniques, based on the desired signal $s(t)$.

Receiver Operating Characteristic (ROC) curves for each desired signal are shown in Fig. 5. The data is based on Monte Carlo experiments with $a = 100$ and $\sigma_\eta^2 = 1$. The ‘Ideal’ curve is the theoretical ROC obtained when directly computing the integral in (8).

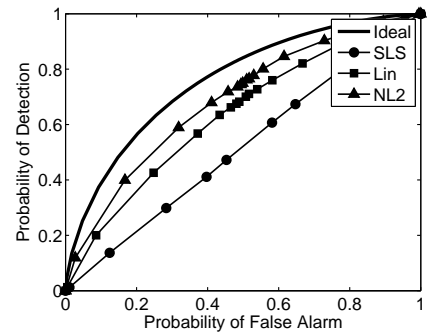
The bilinear performance conforms to the intuition gained in Sections 3 and 4. Specifically, the detection for $s_1(t)$ is close to ideal with only 5 DT multiplies. Conversely, the bilinear approximations are not as good for $s_2(t)$ and $s_3(t)$, which do not have energy concentrated early in time. As seen in Fig. 5, the performance is not ideal, despite the increased number of multiplies.

In contrast, the Nyquist *SLS* method does well only when the signal energy is localized in time and can be captured using a few samples, such as the pulse $s_3(t)$. However, *SLS* does not perform as well as the *NL2* bilinear approximation for any of the desired signals.

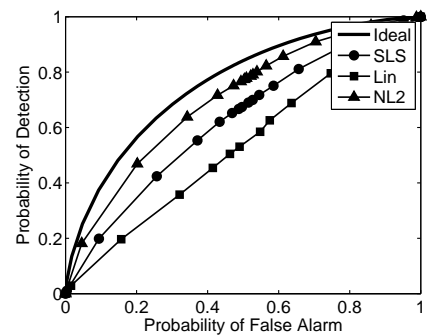
Simulation results based on synthetic signals $s_1(t)$ - $s_3(t)$ indicate that using the bilinear representations may be appropriate in certain



(a) $s_1(t)$, 5 DT Multiplies



(b) $s_2(t)$, 25 DT Multiplies, $T = 0.34$



(c) $s_3(t)$, 25 DT Multiplies, $T = 0.34$

Fig. 5. ROC Curves for the binary detection problem.

binary detection scenarios. The *NL2* performance consistently exceeded that of the Nyquist *SLS* method for a given number of DT multiplies. Furthermore, in applications where CT signals are not appropriately band-limited, using the bilinear representation may be favorable to eliminating signal content via an anti-aliasing filter.

6. REFERENCES

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