Spectral Analysis on a Log-Frequency Scale and the Modeling of Scaling Behavior in Fractal Signals

Gregory W. Wornell and Warren M. Lam
Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139 (617)253-4021

Traditional spectral analysis has been developed primarily in the context of stationary time series, and with the notion of applying such analysis with a fixed frequency resolution over the spectrum. This has led to a wide variety of practical FFT-based algorithms for performing such spectral analysis. However, spectral analysis is also important in modeling and characterizing some important classes of nonstationary signals, including a broad range of fractal processes, *i.e.*, processes in which there is underlying scaling behavior. Furthermore, in many contexts involving nonstationary and/or stationary signals, a spectral analysis is desired in which there is a fixed *percentage* frequency resolution. A rather natural and framework for developing such constant-Q spectral analysis arises out of recently developed wavelet theory, and practical algorithms can be developed exploiting the computational efficiency of the discrete wavelet transform (DWT).

The class of signals of interest in this work possess the characteristic property that when filtered with an ideal bandpass filter whose passband is $\omega_0 \leq \omega \leq \omega_1$, where $0 < \omega_0 \leq \omega_1 < \infty$ are arbitrary, the resulting process is stationary. While this class clearly includes all stationary processes, it also includes some important nonstationary processes. For example, as developed in [1] [2], this class includes the $1/f$ family of fractal processes $x(t)$ which are statistically scale-invariant, *i.e.*, for all $a > 0$,

$$x(t) \stackrel{\mathcal{P}}{=} a^{-H}\, x(at)$$

where $\stackrel{\mathcal{P}}{=}$ denotes equality in a statistical sense.

In the case of $1/f$ processes, such scaling behavior gives rise to a spectrum of the form

$$S_x(\omega) \sim \frac{\sigma^2}{|\omega|^\gamma} \tag{1}$$

through the spectral window of the arbitrary ideal bandpass filter, where $\gamma = 2H + 1$. The suitability of constant-Q spectral analysis and modeling for these processes in particular is well-understood, and, accordingly, the power-law spectra for these processes are typically depicted on a log-log (Bode) plot.

The $1/f$ processes and their corresponding spectra (1) are appropriate for modeling fractal phenomena dominated by a single scaling mechanism over all scales. In practice, however, physical fractal phenomena typically exhibit scaling behavior which is different over different ranges of scales (see, *e.g.*, the seafloor modeling of Goff and Jordan [3]). In such phenomena, there are generally a collection of natural scaling mechanisms in effect, each of which predominates in different scale

This work has been supported in part by the Advanced Research Projects Agency monitored by ONR under Contract No. N00014-89-J-1489, and the Air Force Office of Scientific Research under under Grant No. AFOSR-91-0034.

regime. Spectral analyses of these generalized fractal processes reveal different power-law characteristics in different frequency bands, and Bode plots of the resulting spectra are well-modeled as piecewise linear.

Orthonormal wavelet bases are, in many respects, ideally suited for spectral analysis of the processes described above. In particular, we will show, under relatively mild conditions on the analyzing wavelet basis, that the wavelet coefficients of such processes are effectively uncorrelated, and that the scale-to-scale variance progression of the coefficients reveals the details of the scaling behavior in the underlying processes.

In general, the piecewise linear "Bode" spectra associated with these generalized fractal processes can be effectively modeled using cascade, parallel, and superposition combinations of simpler fractal processes with spectra of the form

$$S_x(\omega) \sim \frac{\sigma^2}{|\omega|^\gamma + p^2}.$$

In addition, as we will show, suitable wavelet analyses of these simpler processes yield effectively uncorrelated coefficients whose scale-to-scale variance progression is of the form

$$\mathrm{Var}\, x_n^m \sim \frac{\tilde{\sigma}^2}{2^{-\gamma m} + \tilde{p}^2}$$

for some $\tilde{\sigma}^2$ and $\tilde{p}$ which depend on σ^2 and p.

Exploiting the wavelet-based framework described above, and using both Maximum Likelihood (ML) and Maximum Spectral Flatness (MSF) criteria, we develop a collection of signal processing algorithms for addressing problems of both fractal signal parameterization and modeling, and fractal signal smoothing and separation. These computationally efficient data-driven algorithms, of which those in [4] [2] [1] constitute a special case, appear to be practical and robust, and we demonstrate results of their application to a variety of both simulated and real data.

References

[1] G. W. Wornell, "Synthesis, analysis, and processing of fractal signals," RLE Tech. Rep. No. 566, M. I. T., Cambridge, MA, Oct. 1991.

[2] G. W. Wornell, "Wavelet-based representations for the $1/f$ family of fractal processes." Submitted to *Proc. IEEE*, Jan. 1992.

[3] J. A. Goff and T. H. Jordan, "Stochastic modeling of seafloor morphology: Inversion of sea beam data for second-order statistics," *J. Geophys. Res.*, vol. 93, pp. 13589–13608, Nov. 1988.

[4] G. W. Wornell and A. V. Oppenheim, "Estimation of fractal signals from noisy measurements using wavelets," *IEEE Trans. Signal Processing*, vol. 40, no. 3, pp. 611–623, 1992.