SIGNAL RECONSTRUCTION FROM ONE BIT OF FOURIER TRANSFORM PHASE*

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ABSTRACT

In this paper, we present new results on the reconstruction of signals from one bit of Fourier transform phase, defined as the sign of the real part of the Fourier transform. Specifically, we develop a new theoretical result which shows that most two-dimensional signals can in fact be reconstructed to within a scale factor from only one bit of FT phase. Experimental results showing images reconstructed from one bit of FT phase are also presented.

I. INTRODUCTION

Signal reconstruction from partial Fourier domain information has been of interest to a number of different authors both for particular applications and for its inherent theoretical value [1]. Previous work in this area has involved signal reconstruction from the FT magnitude or phase [2][3][4] or signed-magnitude [5]. This paper introduces new results on the reconstruction of signals from only one bit of the Fourier transform phase, without any magnitude information. We define "one bit of Fourier Transform phase" to mean the sign of the real part of the Fourier transform, i.e.,

$$S_x(\omega) = \begin{cases} 1 & \text{if } \mathrm{Re}\,\{X(\omega)\} \geq 0 \\ -1 & \text{otherwise} \end{cases} \qquad (1)$$

Using images, we have previously shown [1] that if the correct one bit of phase is combined with a unity or average magnitude, the resulting image maintains many of the features of the original image, and in fact, is identical to a phase-only image formed from the even (symmetric) component of the signal. Furthermore, if an image is synthesized from the correct magnitude and phase with the most significant bit of the phase randomized, the result is unintelligible. These results suggest that the most significant bit of the phase contains much of the information about the nature of the signal. In the next section, we present new theoretical results on reconstruction of a signal from one bit of FT phase. Experimental results showing example images reconstructed from one bit of FT phase are included in section III.

II. THEORETICAL RESULTS

The one bit of phase information is sufficient to reconstruct one-dimensional signals only in a few special cases [1]. However, we have recently developed a new result which shows that one bit of phase is sufficient to reconstruct most two-dimensional signals. In this section, we develop the conditions required to make this reconstruction possible and show that these conditions include a broad class of signals.

Our reconstruction theorem relies upon a result from algebraic geometry, which we shall state here without proof:

Theorem 1 [6][7]. If $X(z_1,z_2)$ and $Y(z_1,z_2)$ are two-dimensional polynomials of degrees r and s with no common factors of degree > 0, then there are at most rs solutions to the following equations:

$$X(z_1,z_2) = 0 \qquad (2)$$
$$Y(z_1,z_2) = 0$$

Essentially, this theorem places an upper bound on the number of points where two two-dimensional polynomials can both be zero if they do not have a common factor.

In the discussion that follows, assume $x[n_1,n_2]$ and $y[n_1,n_2]$ are two-dimensional signals with finite support over a non-symmetric half-plane (or over one quadrant), and let $x_e[n_1,n_2]$ and $y_e[n_1,n_2]$ denote the corresponding even components, i.e.,

$$x_e[n_1,n_2] = \frac{x[n_1,n_2]+x[-n_1,-n_2]}{2} \qquad (3)$$
$$y_e[n_1,n_2] = \frac{y[n_1,n_2]+y[-n_1,-n_2]}{2}$$

Also, let $X_e(z_1,z_2)$ and $Y_e(z_1,z_2)$ denote the z-transforms of $x_e[n_1,n_2]$ and $y_e[n_1,n_2]$, respectively.

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We next develop the following result:

> *Theorem 2.* If $x[n_1,n_2]$ and $y[n_1,n_2]$ are two-dimensional sequences with support over a finite non-symmetric half-plane, with sign Re $\{X(\omega_1,\omega_2)\}$ = sign Re $\{Y(\omega_1,\omega_2)\}$ for all (ω_1,ω_2), and Re $\{X(\omega_1,\omega_2)\}$ takes on both positive and negative values, and furthermore, $X_e(z_1,z_2)$ and $Y_e(z_1,z_2)$ are nonfactorable, then $x[n_1,n_2] = \alpha y[n_1,n_2]$.

To prove this result, we will assume that sign Re $\{X(\omega_1,\omega_2)\}$ = sign Re $\{Y(\omega_1,\omega_2)\}$ and show that $x[n_1,n_2] = \alpha y[n_1,n_2]$ for some α. Naturally, if Re $\{X(\omega_1,\omega_2)\} > 0$ for all (ω_1,ω_2), then sign Re $\{X(\omega_1,\omega_2)\}$ is not sufficient to reconstruct the original signal. Thus, we will assume that there is some region of the (ω_1,ω_2) plane where Re $\{X(\omega_1,\omega_2)\} > 0$ and another region where Re $\{X(\omega_1,\omega_2)\} < 0$. The boundary between these two regions is a curve where Re $\{X(\omega_1,\omega_2)\} = 0$, or equivalently, $X_e(\omega_1,\omega_2) = 0$. Thus,

$$X_e(z_1,z_2)\big|_{z_1=e^{j\omega_1},\, z_2=e^{j\omega_2}} = 0 \tag{4}$$

if ω_1 and ω_2 are on this curve. Since sign Re $\{X(\omega_1,\omega_2)\}$ = sign Re $\{Y(\omega_1,\omega_2)\}$ for all (ω_1,ω_2), equation (4) also holds for $Y_e(z_1,z_2)$. Thus, we have an infinite set of points where

$$X_e(z_1,z_2) = Y_e(z_1,z_2) = 0 \tag{5}$$

Since $x_e[n_1,n_2]$ is nonzero for positive and negative values of n_1 and n_2, $X_e(z_1,z_2)$ is a polynomial in the variables z_1, z_2, z_1^{-1}, and z_2^{-1}. However, if $x[n_1,n_2]$ and $y[n_1,n_2]$ have finite support, then we can find integers N_1 and N_2 such that $z_1^{N_1} z_2^{N_2} X_e(z_1,z_2)$ and $z_1^{N_1} z_2^{N_2} Y_e(z_1,z_2)$ are polynomials in only z_1 and z_2. Furthermore,

$$z_1^{N_1} z_2^{N_2} X_e(z_1,z_2) = 0 \tag{6}$$

$$z_1^{N_1} z_2^{N_2} Y_e(z_1,z_2) = 0$$

over the curve where Re $\{X(\omega_1,\omega_2)\} = 0$, or in other words, over an infinite set of points. Thus, by Theorem 1, $X_e(z_1,z_2)$ and $Y_e(z_1,z_2)$ must have a common factor. If furthermore, we assume that $X_e(z_1,z_2)$ and $Y_e(z_1,z_2)$ are nonfactorable, then $X_e(z_1,z_2) = \alpha Y_e(z_1,z_2)$, $x_e[n_1,n_2] = \alpha y_e[n_1,n_2]$, and $x[n_1,n_2] = \alpha y[n_1,n_2]$.

Although there is no guarantee that a particular signal would satisfy the constraints of Theorem 2, with some informal arguments, we can show that the probability of a random signal satisfying these conditions very rapidly approaches one as the number of points in the signal approaches infinity. First, consider the condition requiring Re $\{X(\omega_1,\omega_2)\}$ to change sign. If the coefficients of the sequence were random, there would be some small, but finite, chance that the first coefficient, $x[0,0]$, would be greater than the sum of the magnitudes of the others, and thus sign Re $\{X(\omega_1,\omega_2)\}$ = sign $x[0,0]$ for all (ω_1,ω_2). Thus, we cannot claim that Theorem 2 applies to "almost all" sequences, that is, with probability one.

We can, however, argue that the probability of a random sequence satisfying the constraints of Theorem 2 very rapidly approaches one as the number of points in the signal approaches infinity. If we assume $x[n_1,n_2]$ is a first quadrant sequence with support $N_1 \times N_2$ and form the one-dimensional sequence

$$a[n_1 + N_1 n_2] = x[n_1,n_2] \tag{7}$$

and if $A(z)$ has at least one zero outside the unit circle, then $A_R(\omega)$ is guaranteed to have at least one sign change over the interval $(0,\pi)$ [1]. Since $A(\omega)$ is a slice of $X(\omega_1,\omega_2)$, Re $\{X(\omega_1,\omega_2)\}$ must change sign somewhere in the (ω_1,ω_2) plane. If we assume that the zeros of $A(z)$ are equally likely to be inside the unit circle as outside, then the probability of $A(z)$ having at least one zero outside the unit circle is

$$p = 1-(.5)^{N_1 N_2 - 1} \tag{8}$$

since $a[n]$ is $N_1 N_2$ points long. Since this condition is sufficient but not necessary for Re $\{X(\omega_1,\omega_2)\}$ to change sign, p represents a lower bound on the probability that a random two-dimensional sequence would satisfy the constraints of Theorem 2. Furthermore, we note that p approaches one very rapidly even for small images. For example, for a 3×3 image, $p = 0.9961$, and for a 4×4 image, $p = 0.999969$. For a 256×256 image, $p \approx 1-10^{-20000}$.

Finally, note that since "almost all" two-dimensional polynomials are nonfactorable, the nonfactorability constraint is satisfied with probability one [4]. Also, it is possible to modify Theorem 2 to include constraints on each factor instead of requiring nonfactorability, although we have not chosen to do so here.

III. EXPERIMENTAL RESULTS

A common algorithm used in signal reconstruction problems is an iterative one which alternately imposes constraints in the time and frequency domains. This algorithm can be applied to the problem of reconstruction from one bit of phase by imposing the correct sign of the real part of the FT in the frequency domain and imposing the known region of support in the time domain. Unfortunately, since knowledge of the exact points of discontinuity is necessary for the signal to be uniquely specified, no algorithm which uses only the sign of the real part of the DFT points can be guaranteed to converge to the correct sequence. Experimentally, we have found that this method will converge to a sequence which approximates the original signal if the DFT size used is at least 4 times the size of the original signal. An example is shown in Figure 1, where we show the original image (a) and the image reconstructed from one bit of FT phase (b). In this example, the original image is 64×64 points, 256×256 DFTs were used, and the results shown were obtained with 25 iterations, accelerated as discussed in [3] and [4].

Another reconstruction algorithm involves solving a set of linear equations of the form:

$$\sum x[n_1,n_2] \cos(\omega_1 n_1+\omega_2 n_2) = 0 \qquad (9)$$

where each equation uses a different pair of frequencies (ω_1,ω_2) for which the equality is known to hold. (We substitute $x[0,0] = 1$ in order to obtain a non-zero solution.) We have found this method to be particularly sensitive to numerical errors in the values of (ω_1,ω_2). However, if the values of ω_1 and ω_2 are obtained to four- or five-digit accuracy and if the number of equations used is greater than the number of unknowns and a least-squares solution is obtained, results indistinguishable from the original signal can be achieved. An example is shown in Figure 2, which shows the original image (a) and the image reconstructed by solving the above equations (b). In this example, the original image is 13×8 (104 points) and 106 equations in 103 unknowns were used. We have used this procedure to reconstruct a number of different images of varying sizes which satisfy the constraints of Theorem 2, and have always successfully recovered the original image provided enough equations were used.

IV. CONCLUSIONS

In this paper, we have developed conditions under which two-dimensional signals can be reconstructed from one bit of FT phase. We have also shown that these conditions apply to a broad class of signals by showing that if the coefficients of the signal are random, the probability of a sequence satisfying these conditions very rapidly approaches one as the number of points in the signal approaches infinity. We have also included examples of images reconstructed from one bit of FT phase.

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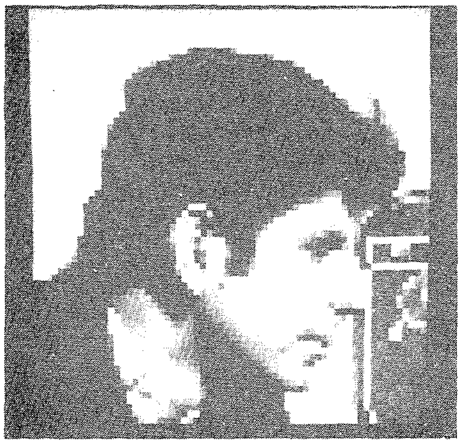


Fig. 1 (a) original image

(b) image reconstructed with iterative algorithm

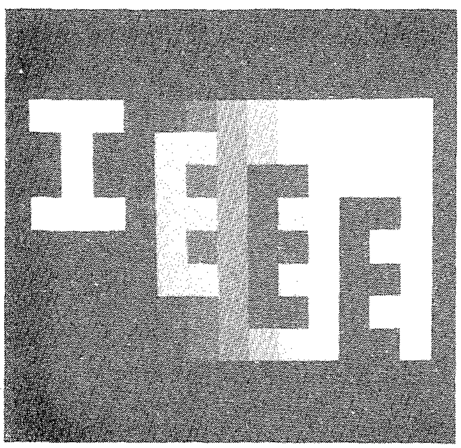
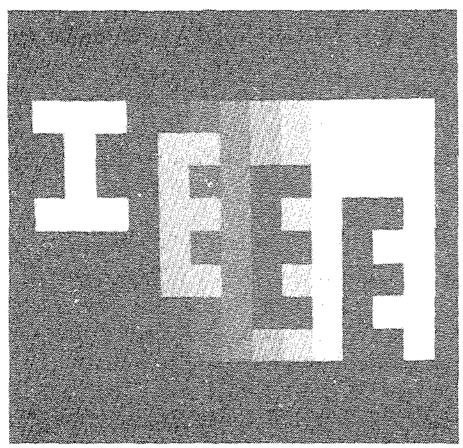


Fig. 2 (a) original image



(b) image reconstructed by solving linear equations