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Reconstruction of multidimensional signals from zero crossings

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In this paper we present and review recent results that we have developed on the reconstruction of multidimensional signals from zero-crossing or threshold-crossing information. Specifically, we develop theoretical results that state conditions under which multidimensional band-limited signals are uniquely specified to within a scale factor with this information. We also present an algorithm for recovering signals from zero-crossing or threshold-crossing information once it is known that the signals satisfy the appropriate constraints. Examples showing images recovered from threshold crossings are included.

1. INTRODUCTION

Over a period of years, there has been a significant amount of interest in the problem of representing signals with zero crossings. The great majority of research in this area has been in communication theory and has concentrated on one-dimensional signals, although recently extensions to two-dimensional signals have also been reported.[1–4] In this paper we present and review recent results that we have developed on the reconstruction of multidimensional signals from zero-crossing information. These results are much less restrictive and appear to be more broadly applicable than results based on two-dimensional extensions of one-dimensional results.

The importance of zero-crossing locations in determining the nature of both one- and two-dimensional signals has been recognized for some time. Experiments in speech processing have shown that speech with only the zero-crossing information preserved (hard-clipped speech) retains much of the intelligibility of the original speech.[5] Also, a wide variety of papers in image processing and vision stress the importance of the information contained in the edges of objects, and one theory of human vision relies primarily on edge detection as the mechanism by which humans process visual information.[3]

There are also a variety of other types of applications in which the zero crossings or threshold crossings are available and it is desired to recover the original signal. One possible application occurs when an image is clipped or otherwise distorted in such a way as to preserve zero-crossing or level-crossing information and it is desired to recover the original signal from this information. This might happen if an image is recorded on a high-contrast film or, more generally, on film with an unknown nonlinear monotonic gray-scale distortion. If it is possible to recover the original signal from its threshold crossings, then it is possible, at least in principle, to recover the original signal from its distorted version and to determine the type of nonlinearity present. In addition, it is not necessary for the nonlinearity to be monotonic over its entire range; it is necessary only that the chosen threshold on the distorted signal correspond to a unique threshold on the original signal. This could potentially be useful in an application such as medical archiving, in which intensity levels of images recorded on film are likely to become distorted over time but threshold-crossing information could be preserved. In some archiving applications it is unlikely that any particular image may need to be retrieved, but it is important to be able to recover the image if necessary even if the process is expensive or time consuming. Another possible type of application of results on reconstruction from zero-crossing information is in a variety of design problems such as filter design[6] and antenna design.[7] In these cases, one could potentially specify the zero-crossing points or null points of the filter response or antenna pattern and then use these points to derive the remainder of the response.

One might also consider the possibility of exploiting the information in threshold crossings for signal coding and data compression. However, in representing a two-dimensional signal with zero crossings or threshold crossings, it is important to recognize that the amplitude information in the original signal is embedded in the exact location of the threshold crossings. Consequently, while the original signal can be sampled at the Nyquist rate, the threshold-crossing representation may require a considerably higher, possibly infinite, sampling rate to preserve the threshold-crossing locations adequately. Thus the total number of bits or bandwidth required in the threshold-crossing representation might well be higher than that required by sampling and quantizing the original signal. For this reason, we expect results on signal reconstruction from threshold crossings to be more useful in applications in which the exact threshold-crossing points are available. It is possible, however, to view

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the representation of signals with threshold crossings as a potential trade-off between the bandwidth and the dynamic range necessary for transmitting a signal. If the available bandwidth is sufficient to preserve the threshold-crossing locations accurately, then the dynamic-range requirements might be greatly reduced if the signal could be recovered from the threshold-crossing locations.

In Section 2 we review a number of known results that state conditions under which one- and two-dimensional signals are uniquely specified with zero crossings. In Section 3 we develop our basic result on the unique representation of periodic two-dimensional signals with zero crossings and discuss a number of extensions, including the extension to crossings of a threshold other than zero, signals with dimension higher than two, and nonperiodic signals. In Section 4 we present a simple algorithm for recovering signals from zero crossings or threshold crossings and show some example images that we have recovered from this information.

2. RELATED RESEARCH

A number of papers in communication theory have dealt with the question of recovering a one-dimensional signal from its zero crossings. (A more detailed review of this work can be found in Ref. 8.) These results are primarily based on the fact that a band-limited function is entire and thus is almost completely specified by its zeros (real and complex). A band-limited signal is uniquely specified by its (real) zero crossings only if all its zeros are guaranteed to be real. Thus a number of previous research efforts concentrated on identifying conditions under which signals have only real zeros and on developing methods for modifying a signal so that all its zeros become real. One result in this area is that a one-dimensional complex signal with no energy for negative frequencies is uniquely specified by the zero crossings of its real part if the complex signal has zeros only in the upper half-plane.[8,9] (A more general form of this result is given in Ref. 10.) One method of modifying signals so that all their zeros become real is to add a sinusoid of sufficient amplitude at a frequency corresponding to the band edge[11]; another is to differentiate the signal repeatedly.[9] Some modulation schemes have also been shown to produce only signals with real zeros.[12] Fairly recently, in response to experimental results presented by Voelcker and Requicha,[13] Logan[10] developed a new class of bandpass signals that are uniquely specified by their zero crossings. Specifically, Logan showed that a signal with a bandwidth of less than one octave is uniquely specified by its zero crossings if it has no zeros in common with its Hilbert transform other than real simple zeros. This means that almost all bandpass signals of bandwidth less than one octave are uniquely specified by their zero crossings. It is also possible to interpret results on unique specification of signals with zero crossings as a type of sampling, in which the samples consist of the set of points (times) corresponding to zero crossings, as opposed to the amplitude of the signal at particular fixed instants.[8] Using this point of view, sampling might consist of adding a sine wave at the appropriate frequency and recording those instants when the resulting signal crosses zero or, equivalently, recording those instants where the original signal crosses a sinusoid. Interpolation would then consist of generating a signal with sine-wave crossings at the specified instants. Sampling and interpolation systems using this approach have been designed, implemented, and found to produce good results.[14]

Despite the number of results on the unique specification of signals with zero crossings, most one-dimensional band-limited signals encountered in practice do not satisfy the constraints associated with any of the above and are not uniquely specified by their zero crossings unless they satisfy some additional constraints that effectively guarantee that they contain a sufficient number of zero crossings. In fact, it has been shown[15] that almost all sample functions of a band-limited Gaussian random process are not uniquely specified by zero crossings.

Although a considerable amount of theoretical work has been devoted to the problem of unique representation of one-dimensional signals with zero crossings, much less work has been devoted to the corresponding two-dimensional problem. Logan's result has been extended to two dimensions[3,4] by requiring a one-dimensional signal derived from the original two-dimensional signal to satisfy the constraints of Logan's theorem. In addition, one-dimensional results on unique specification with sine-wave crossings have been extended to two-dimensional problems.[16] However, the two-dimensional problem is fundamentally different from the one-dimensional problem since the zero crossings are actually zero-crossing contours and not isolated points, as in the one-dimensional case. This difference allow us to specify uniquely a two-dimensional signal with zero crossings under much less severe restrictions than are necessary for one-dimensional signals. The difference can be easily appreciated by thinking of the representation of signals in terms of zero crossings as a form of nonuniform sampling, with each zero-crossing point representing one sample. In one dimension, each zero-crossing point corresponds to one sample of the signal, and the zero crossings are sufficient for unique representation of the signal only if the zero-crossing rate is high enough.[15] In two dimensions, each zero-crossing contour corresponds to an infinite number of samples of the signal. Thus it is reasonable to suggest that two-dimensional signal may be specified with zero crossings under more general conditions than those required for a one-dimensional signal. This is in fact true, and these results will be presented in Section 3.

3. THEORETICAL RESULTS

In this section we present theoretical results that we have developed on unique specification of multidimensional signals with zero crossings. We will begin by discussing our basic result on the unique specification of band-limited, periodic, two-dimensional signals with zero crossings. Our results are simpler to develop for periodic signals than for arbitrary signals since we can represent these signals as polynomials in a Fourier-series representation and apply well-known results on polynomials from algebraic geometry. We will then discuss a variety of extensions to this result including the extension to signals of dimension higher than two and to nonperiodic signals.

A. Two-Dimensional Periodic Signals

To develop our basic result on the unique specification of two-dimensional, periodic signals with zero crossings, con-

sider a real, band-limited, continuous-time, periodic signal $f(x, y)$ with periods T_1 and T_2 in the x and y directions, respectively. We can express $f(x, y)$ as a polynomial using the Fourier series representation:

$$f(x, y) = \sum_{n_1}\sum_{n_2} F(n_1, n_2) W_1^{n_1} W_2^{n_2}, \quad (1)$$

where

$$W_1 = \exp[j(2\pi x/T_1)],$$

$$W_2 = \exp[j(2\pi x/T_2)].$$

The coefficients $F(n_1, n_2)$ are the Fourier-series coefficients and represent the spectrum of $f(x, y)$. Since we are assuming $f(x, y)$ to be band limited, the sums in Eq. (1) must be finite. The set of points (n_1, n_2) outside which $F(n_1, n_2)$ is constrained to be zero is referred to as the region of support of the spectrum. Assume that $F(n_1, n_2) = 0$ outside the region $-N_1 \le n_1 \le N_1, -N_2 \le n_2 \le N_2$. To modify Eq. (1) to have the form of a two-dimensional polynomial, we can write

$$f'(x, y) = W_1^{N_1} W_2^{N_2} f(x, y)$$

$$= \sum_{n_1=0}^{n_1=2N_1} \sum_{n_2=0}^{n_2=2N_2} F(n_1 - N_1, n_2 - N_2) W_1^{n_1} W_2^{n_2}. \quad (2)$$

Although in the discussion that follows we shall refer to the representation of $f(x, y)$ as a Fourier-series polynomial, it should be kept in mind that, strictly speaking, we are referring to the representation of the modulated signal $f'(x, y)$ in Eq. (2) as a polynomial.

With the signal represented as a polynomial, we will use a well-established result on polynomials in two variables to develop our results on the unique specification of signals with zero crossings. We will state the basic result on two-dimensional polynomials here without proof; the detailed proof is available in Refs. 17 and 18 as well as in a number of other texts on algebraic geometry.

Theorem 1. If $p(x, y)$ and $q(x, y)$ are two-dimensional polynomials of degrees r and s with no common factors, then there are at most rs distinct pairs (x, y), where

$$p(x, y) = 0$$

and

$$q(x, y) = 0. \quad (3)$$

In this theorem, the degree of a polynomial in two variables is defined as the sum of the degrees in each variable (for each term), that is, the degree of a two-dimensional polynomial $p(x, y)$ is equivalent to the degree of the one-dimensional polynomial $p(x, x)$. The rs distinct pairs (x, y) described in this theorem consist of rs points anywhere in the complex (x, y) plane. Essentially, theorem 1 places an upper bound on the number of points where two two-dimensional polynomials can both be zero if they do not have a common factor.

A stronger form of theorem 1 is available that guarantees that the zero sets of polynomials intersect in exactly rs points rather than simply stating an upper bound. This stronger result, referred to as Bezout's theorem in algebraic geometry, requires including the multiplicity of intersections as well as points that lie at infinity (e.g., two parallel lines are considered to intersect in one point at infinity). Bezout's theorem can be thought of as a generalization of the fundamental theorem of algebra, which guarantees that a one-dimensional nth-degree polynomial has exactly n roots, provided that multiplicity is included.

As we shall discuss in more detail later, theorem 1 implies that a nonfactorable two-dimensional polynomial of degree d is uniquely specified to within a scale factor by $d^2 + 1$ zero-crossing points. Therefore a polynomial of degree N in each variable (i.e., $d = 2N$) requires at most $4N^2 + 1$ zero crossings. Since a polynomial of degree N in each variable will have $(N + 1)^2$ coefficients, one might expect the polynomial to be uniquely specified to within a scale factor with $(N + 1)^2 - 1$ distinct points where it is zero. A consequence of theorem 1 is that this set is *not* guaranteed to be sufficient, but any set of $4N^2 + 1$ distinct points *is* guaranteed to be sufficient.

1. *Basic Result*

We use the representation of signals as polynomials and the result on intersection of zero sets of polynomials to establish our primary result on the unique specification of periodic signals with zero crossings. Several extensions to this result are presented in Subsection 3.A.2.

To see how results on the intersection of curves apply to the problem of unique specification of two-dimensional signals with zero crossings, consider a real, band-limited, periodic signal $f(x, y)$ expressed as a polynomial in the Fourier-series representation in Eq. (2). We assume that there are some regions where $f(x, y)$ is positive and some regions where $f(x, y)$ is negative. These regions are separated from each other by a contour where $f(x, y) = 0$. If another signal $g(x, y)$ has the same zero-crossing contours as $f(x, y)$, then there are an infinite number of points where both $f(x, y)$ and $g(x, y)$ are zero. We can then use theorem 1 to show that $f(x, y)$ and $g(x, y)$ must have a common factor. If, furthermore, we know that $f(x, y)$ and $g(x, y)$ are irreducible when expressed as polynomials, as in Eq. (2), then they must be equal to within a scale factor. The result can be stated as follows:

Theorem 2. Let $f(x, y)$ and $g(x, y)$ be real, two-dimensional, doubly periodic, band-limited functions with sign $f(x, y)$ = sign $g(x, y)$, where $f(x, y)$ takes on both positive and negative values. If $f(x, y)$ and $g(x, y)$ are nonfactorable when expressed as polynomials in the Fourier-series representation (2), then $f(x, y) = cg(x, y)$.

Proof. We will prove this result by starting with two signals $f(x, y)$ and $g(x, y)$ that satisfy the constraints of the theorem and showing that they must be equal to within a scale factor. Since we know that $f(x, y)$ takes on positive and negative values, there must be some region of the (x, y) plane where $f(x, y) > 0$ and another region where $f(x, y) < 0$. Since $f(x, y)$ is band limited and therefore continuous, the boundary between these regions is a contour where $f(x, y) = 0$. Since sign $f(x, y)$ = sign $g(x, y)$ for all (x, y), the same arguments also hold for $g(x, y)$. Thus we have contours in the x, y plane where

$$f(x, y) = g(x, y) = 0. \quad (4)$$

Also, if N_1 and N_2 are defined as in Eq. (2), we have

$$W_1^{N_1} W_2^{N_2} f(x, y) = 0,$$
$$W_1^{N_1} W_2^{N_2} g(x, y) = 0 \quad (5)$$

over these contours. Thus we have an infinite set of points where two polynomials in the variables W_1, W_2 are known to be zero. Thus, by theorem 1, $f(x, y)$ and $g(x, y)$ must have a common factor. If, furthermore, we assume that $f(x, y)$ and $g(x, y)$ are nonfactorable when expressed as polynomials in Eq. (2), then $f(x, y) = cg(x, y)$.

Note that, in order to satisfy theorem 1, it is not necessary to know the location of all the zero-crossing contours; it is necessary only to know the location of a sufficient number of points along these contours. Thus, in theory, any zero-crossing contour in the (x, y) plane is sufficient to specify the signal uniquely (since it contains an infinite number of points) even if the region where $f(x, y) < 0$ is very small. It is also possible to sample the zero-crossing contours, i.e., to specify the signal uniquely with only a finite set of discrete points from the zero-crossing contours. This possibility will be explored in more detail in Subsection 3.A.2.

Having established a set of conditions that guarantees that a signal is uniquely specified by some partial information, it is worthwhile to determine whether these conditions are likely to apply to a typical signal encountered in practice. First, we note that the irreducibility constraint is satisfied with probability one, since it has been shown that the set of reducible m-dimensional polynomials forms a set of measure zero in the set of all m-dimensional polynomials (for $m > 1$)[19] and that this set is an algebraic set.[20] The more restrictive constraint is the constraint requiring the signal to be strictly band limited. Although signals encountered in practice are generally not strictly band limited, in many applications signals are commonly assumed to be band limited, and furthermore it is common to low-pass filter signals when necessary for particular processing techniques. Another conceivable difficulty with this result is that in some applications, such as image processing, the signals are constrained to be positive and thus will not contain zero crossings. This problem will be eliminated in the following subsection when we extend this result to include crossings of an arbitrary threshold instead of just zero crossings.

2. Extensions

Although theorem 2 states a number of conditions under which a signal is uniquely specified with its zero crossings, it is also possible to develop a number of variations or extensions of this result. All the extensions developed in Ref. 2 for the case of reconstructing finite-length signals from Fourier-domain zero crossings also apply directly to this problem. In this subsection, we will review some of the more important extensions.

Finite-Length Signals. The fact that knowledge of all the zero contours in the (x, y) plane is not necessary to specify the signal uniquely allows us to extend this result to signals that are not periodic but are finite length. This extension is important since most signals encountered in practice are finite length. Consider the case in which $f(x, y)$ is a finite segment of a periodic signal satisfying the constraints of theorem 2. For example, if $f(x, y)$ represents one period of a band-limited periodic function $\tilde{f}(x, y)$:

$$\tilde{f}(x, y) = \sum_{n_1} \sum_{n_2} f(x + n_1 T_1, y + n_2 T_2), \quad (6)$$

then it is possible to recover $f(x, y)$ from its zero crossings, provided that $\tilde{f}(x, y)$ satisfies the constraints of theorem 2, even though $f(x, y)$ itself is not band limited. More generally, it is not necessary for the duration of $f(x, y)$ to be equal to one period of the corresponding periodic function. Thus $f(x, y)$ can represent a finite segment of a variety of different periodic functions. In order for $f(x, y)$ to be uniquely specified by its zero crossings, we need only one periodic function containing $f(x, y)$ to be band limited.

Threshold Crossings. It is possible to generalize the results presented above to allow the signals to be specified by crossings of an arbitrary threshold rather than simply zero crossings. This is important in applications such as image processing in which signals represent energy or intensity and thus are constrained to be positive. These signals contain no zero crossings but may contain points (contours) where the signal crosses a particular threshold. More generally, it is possible to allow crossings of any known band-limited periodic function. The basis for these extensions is relatively straightforward. Specifically, by subtracting the known band-limited periodic function $h(x, y)$ from the signal $f(x, y)$, we create a new band-limited signal $g(x, y)$. The zero crossings of $g(x, y)$ correspond to the contours where $f(x, y)$ crosses $h(x, y)$. In the special case in which $h(x, y)$ is a constant, the zero crossings of $g(x, y)$ are the threshold crossings of $f(x, y)$. While this extension may seem obvious, it is important to recognize that it is not possible to extend Logan's theorem (and many other one-dimensional results) to permit crossings of an arbitrary threshold. Thus the possibility of such an extension provides an important distinction between our work and earlier work with one-dimensional signals.

Discrete Zero-Crossing Points. As mentioned earlier, it is possible to state theorem 2 in a slightly different way so that it is possible to specify a signal uniquely with a finite set of discrete zero-crossing points, essentially allowing us to sample the zero-crossing contours. This result is important since any practical algorithm for recovering signals from zero-crossing information can make use of only a finite number of zero-crossing points. Let us first emphasize that we are referring to sampling the zero-crossing locations *along a zero-crossing contour*, not to sampling of the sign of the original signal at each point on a predetermined grid. This is distinct from the type of sampling used in many signal-processing problems in which signals are specified with samples over a particular grid. The difficulty with sampling the sign information is that the information necessary to apply our results to specify a signal uniquely is contained in the exact location of the zero crossings, and this information is lost when [sign $f(x, y)$] is sampled. In practical applications, of course, it may be possible for a signal to be represented to sufficient accuracy with a finite set of samples of [sign $f(x, y)$].

Since theorem 1 specifies the number of points where two two-dimensional polynomials can both be zero, we can use this theorem to establish that a particular number of arbitrarily chosen zero-crossing points is guaranteed to be sufficient for unique specification. The exact number of zero-crossing points sufficient for unique specification depends

on the size and shape of the spectrum of the signal. We will state our results in terms of rectangular spectra since these shapes are common in applications and are straightforward to understand. The result could be easily modified for spectra of different shapes or could be applied directly to a problem involving a different spectrum by simply assuming a rectangular region large enough to enclose the actual region. If reference to a region of support $R(N)$ specifies that the spectrum is zero outside the region $-N \leq n_1, n_2, \leq N$, then we can state the following:

Theorem 3. Let $f(x, y)$ and $g(x, y)$ be real, two-dimensional, doubly periodic, band-limited functions with a spectrum with region of support $R(N)$. If $f(x, y)$ and $g(x, y)$ are nonfactorable when expressed as polynomials in the Fourier-series representation [Eq. (2)], and $f(x, y) = g(x, y) = 0$ at more than $16N^2$ distinct points in one period, then $f(x, y) = cg(x, y)$ for some real constant c.

Proof. Recall that the proof of theorem 2 requires stating that two polynomials $W_1^{N_1}W_2^{N_2}f(x, y)$ and $W_1^{N_1}W_2^{N_2}g(x, y)$ are equal to within a scale factor, given that they are both zero at an infinite number of points. Substituting $N_1 = N_2 = N$ in the case of theorem 3, we know that $W_1^N W_2^N f(x, y) = W_1^N W_2^N g(x, y) = 0$ at more than $16N^2$ points in one period, that is, at more than $16N^2$ distinct values of the variables (W_1, W_2). These polynomials are of degree $4N$ and thus, by theorem 1, can have at most $16N^2$ common zeros. Thus $W_1^N W_2^N f(x, y) \equiv cW_1^N W_2^N g(x, y)$ and the theorem follows.

Although we have shown that any $16N^2 + 1$ zero-crossing points are sufficient for unique representation of a signal under the constraints of the theorem, we have not shown that all $16N^2 + 1$ zero-crossing points are *necessary* for unique representation. In fact, for the particular case of a spectrum with rectangular region of support, Zakhor and Izraelevitz[21] have shown that a two-dimensional periodic signal is uniquely represented with any $8N^2 + 1$ zero-crossing points (under the same constraints as theorem 3) by developing a new result similar to theorem 1 that applies when the polynomials are considered to have a specified degree in each variable, as opposed to a specified total degree. In addition, we have found that it is often possible to represent a signal with a set of $(2N + 1)^2 - 1$ zero-crossing points (the same as the number of unknown Fourier coefficients), although we have found counterexamples that indicate that this is not true for *all* sets of $(2N + 1)^2 - 1$ points. We speculate that, if the $(2N + 1)^2 - 1$ zero-crossing points are chosen randomly, then, with probability one, these points will be sufficient to represent the signal uniquely.

B. Arbitrary Multidimensional Signals

Although up to this point we have been concerned primarily with the representation of two-dimensional periodic signals with zero crossings, it is also possible to develop similar results for signals with dimensions higher than two and for arbitrary nonperiodic signals. The results developed earlier do not apply to these problems because they require us to represent the signal as a polynomial in two variables, which is possible only for periodic two-dimensional signals. Since the mathematics involved in the proofs of these additional results is somewhat involved and the basic concepts are quite similar to those in the two-dimensional periodic case, we shall briefly state our additional results without proof. More details of these results can be found in Ref. 22.

First, consider a multidimensional, periodic signal with dimension greater than two. This signal can be expressed as a polynomial in a Fourier-series representation similar to Eq. (2), using a polynomial of more than two variables. The zero crossings of a signal with dimension higher then two are not contours in a two-dimensional space but are surfaces in a multidimensional space. Theorem 1 cannot be applied directly since it applies only to polynomials of two variables. An extension of theorem 1 is available, although the result is not quite so straightforward as for theorem 1. In general, it is not possible to state that two polynomials in an arbitrary number of variables have common zeros at a finite number of points. However, it is possible to characterize the intersection of two surfaces, each described by a polynomial equation, as another surface with a specified dimension and degree. Using this procedure, we have developed a result similar to theorem 2 for signals with dimensions higher than two. Our result can be stated as follows:

Theorem 4. Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be real, m-dimensional, periodic, band-limited functions with sign $f(\mathbf{x})$ = sign $g(\mathbf{x})$, where $f(\mathbf{x})$ takes on both positive and negative values. If $f(\mathbf{x})$ and $g(\mathbf{x})$ are nonfactorable when expressed as polynomials in the Fourier-series representation [Eq. (2)], then $f(\mathbf{x}) = cg(\mathbf{x})$.

It is also possible to develop additional results that do not require the signals to be periodic. The problem is more difficult mathematically since it is in general not possible to express an arbitrary signal as a polynomial in a Fourier-series representation. Nevertheless, since the zeros of an arbitrary band-limited function constitute an analytic set, it is possible to find corresponding results characterizing the intersection of analytic sets. We have applied this theory to develop results analogous to theorem 2 for arbitrary (nonperiodic) two-dimensional signals. Details of these results can be found in Ref. 23.

4. RECONSTRUCTION

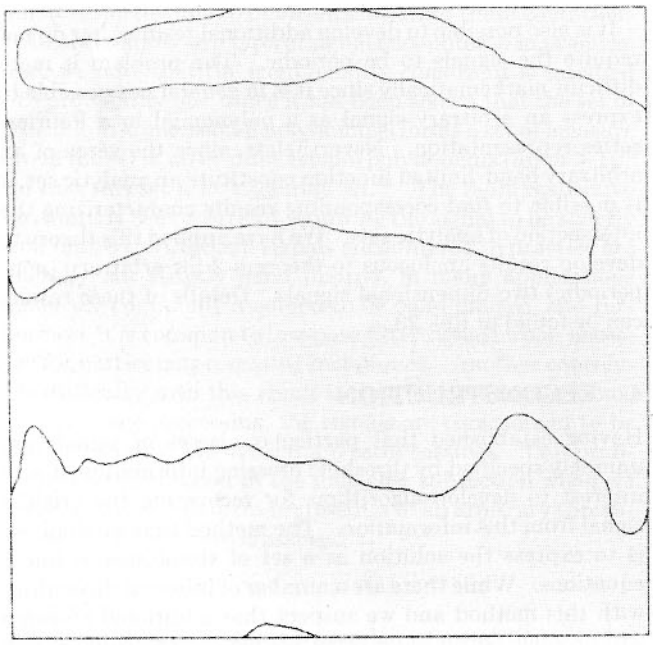
Having established that particular classes of signals are uniquely specified by threshold crossing information, it is of interest to develop algorithms for recovering the original signal from this information. The method that we shall use is to express the solution as a set of simultaneous linear equations. While there are a number of inherent difficulties with this method and we suspect that additional research will produce better algorithms, we have successfully recovered example images by using this algorithm, and it does effectively illustrate problems that occur during the reconstruction.

Our reconstruction algorithm involves first choosing a set of p points where the signal is known to be zero (or known to cross a given threshold) and solving then the following set of equations:

$$\sum_{(n_1,n_2)\in R} F(n_1, n_2) \exp[j(2\pi x_1 n_1)/T_1]\exp[j(2\pi y_1 n_2)/T_2 = 0, \tag{7}$$

where R denotes the known region of support of the spectrum and each equation uses a different pair of points (x_1, y_1) for which the equality is known to hold (i.e., points on the zero-crossing contours). We generally choose p to be great-

(a)



(b)

(c)

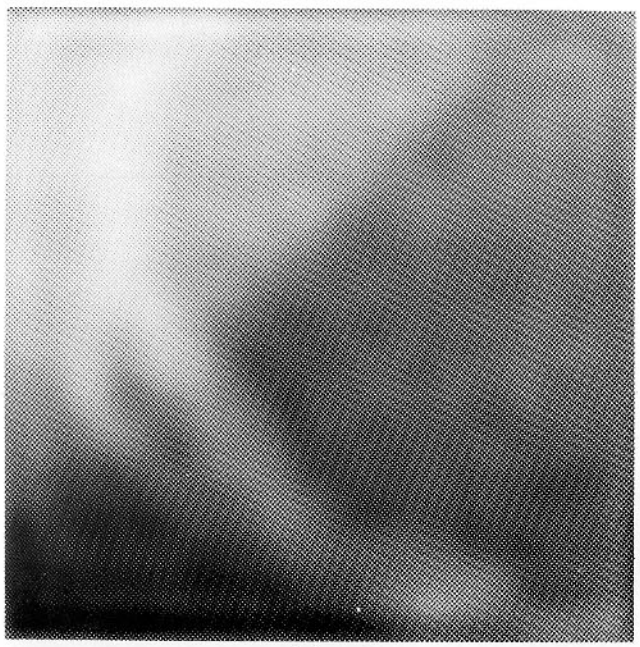
Fig. 1. Reconstruction from zero crossings: (a) original image, (b) threshold crossings of (a), (c) recovered image.

er than the number of unknowns and find a least-squares solution to these equations. There are two reasons for using more equations than unknowns in these problems. First, as mentioned earlier, we cannot guarantee a unique solution to these equations if p is equal to the number of unknown Fourier coefficients, but we can guarantee a unique solution if p is chosen to satisfy theorem 3. [These equations have a unique solution once the scale factor is specified by setting one point to its known value. We have found it simplest to set $F(0, 0)$ to the known mean value of the signal]. Another reason for using more equations than unknowns is to improve the numerical stability of the results, particularly when solving for a large number of unknowns.

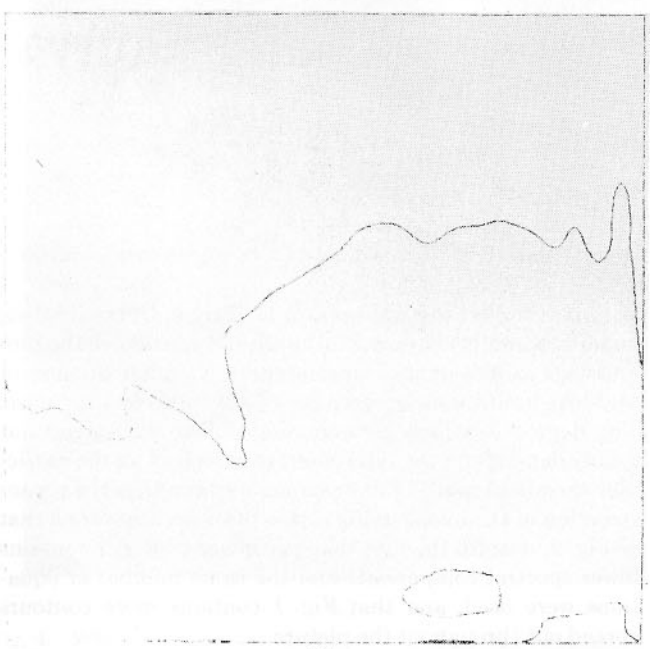
Experimentally, we have found that, when recovering signals with a narrow bandwidth (i.e., a small number of unknown Fourier coefficients), it is often possible to use the same number of equations as unknowns. Additional equations have been necessary only in special cases in which the original zero-crossing points were carefully chosen to correspond to zero-crossing points of a different image as well as the desired image. As mentioned earlier, we speculate that, if the zero-crossing points are chosen randomly, then with probability one it will be possible to use the same number of equations as unknowns as long as the problem is small enough to avoid numerical difficulties.

Examples of two images recovered with this method are given in Figs. 1 and 2, which show the original image [Figs. 1(a) and 2(a)]; an image consisting of the threshold crossings

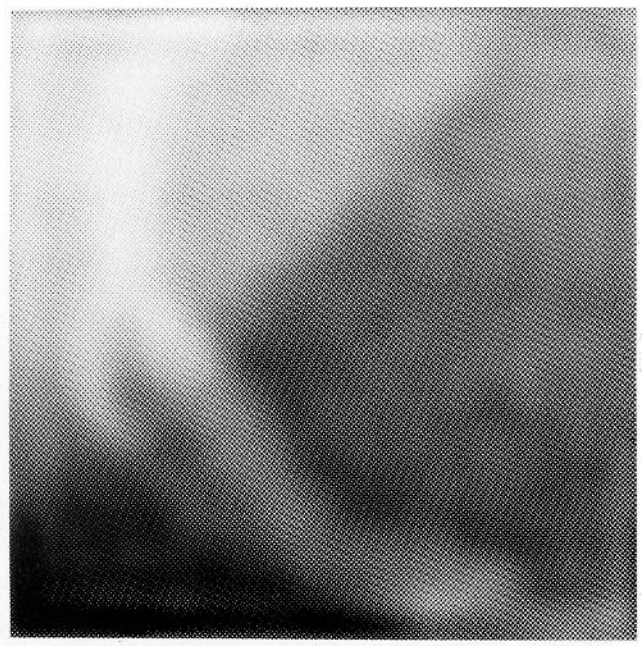
[Figs. 1(b) and 2(b)], i.e., contours showing where the original image crosses a particular threshold; and the image reconstructed by solving the linear equation [Figs. 1(c) and 2(c)]. (Additional examples are given in Refs. 22 and 24.) In these examples, the original images were obtained by low-pass filtering similar images and removing some low-amplitude Fourier-transform points so that it would be practical to solve a set of linear equations for the remaining points. The exact size and shape of the spectrum of the resulting image, i.e., the region of support of the Fourier transform, were then assumed to be known. Precise values of the zero-crossing points were found by taking the discrete Fourier transform of the image, using these coefficients to express the image as a polynomial, as in Eq. (2), and then using a numerical technique to find the zeros of this polynomial to approximately 16-digit accuracy. (Since the image is assumed to be band limited and sampled at the Nyquist rate or higher, it is possible to use the samples of the image to compute the intensity of the image at any desired point between the given picture elements and thus to determine the precise zero-crossing locations.) Equation (7) was then solved by using a QR decomposition[25] and double-precision arithmetic. In the case of Fig. 1, the image contains 228 independent spectral components, a total of 600 equations in 454 unknowns were used (the spectral components are complex and contribute two unknowns), and the normalized rms error (rms error/rms signal) is approximately 0.000065. In the case of Fig. 2, the image has 178 independent spectral components, a total of 600 equations in 354 unknowns were used, and the normalized rms error is approximately 0.027.



(a)



(b)



(c)

Fig. 2. Reconstruction of x ray: (a) original image, (b) threshold crossings of (a), (c) recovered image.

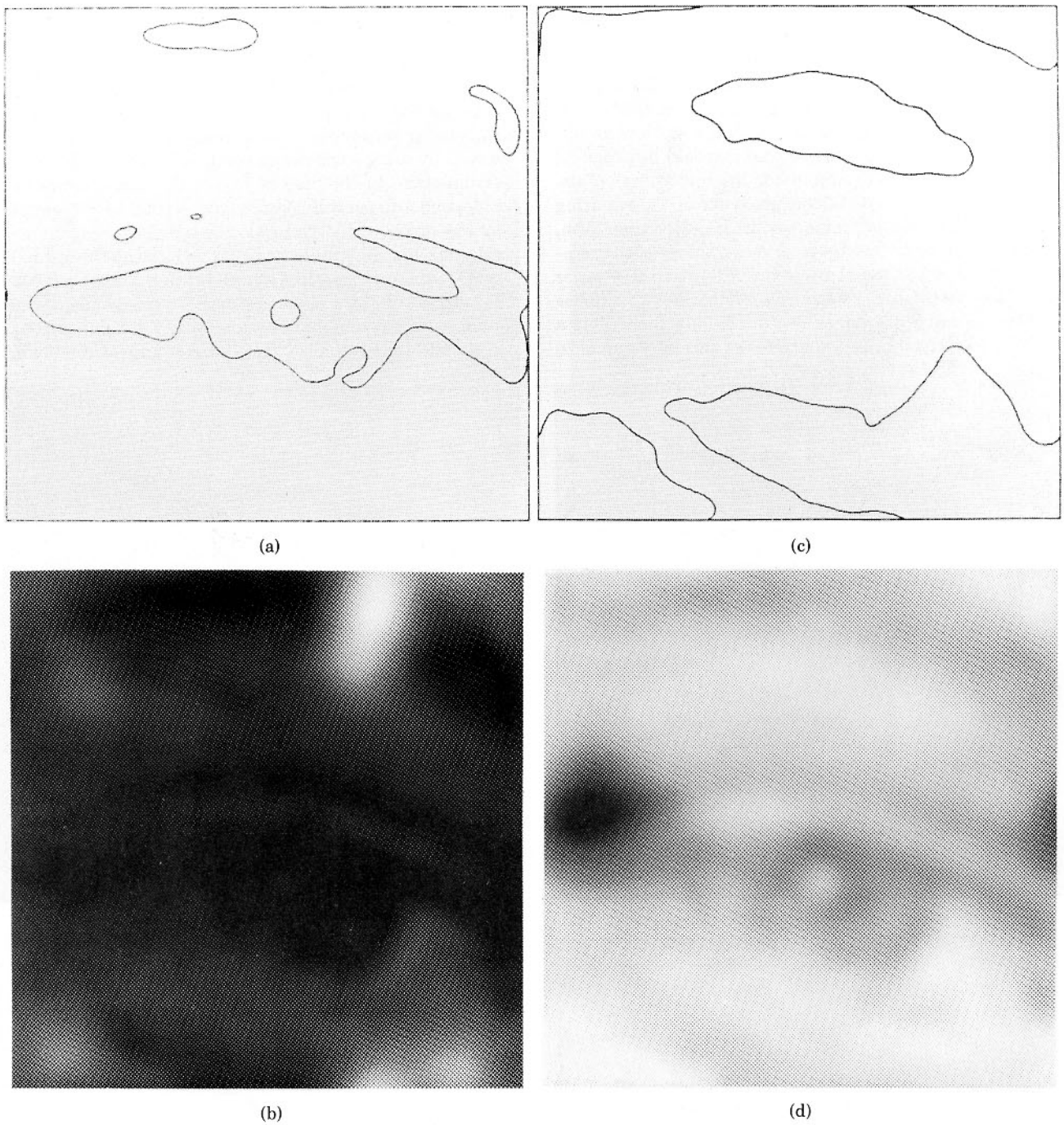


Fig. 3. Effect of different thresholds: (a) threshold, 0.27; (b) image recovered from (a); (c) threshold, 0.64; (d) image recovered from (c).

The entire procedure takes approximately 2 h of CPU time on a VAX 750, although the exact timing depends strongly on the number of equations and the number of unknowns. Roughly one third of this time is spent finding the zero-crossing points, and the remaining two thirds is spent solving the linear equations.

In experimenting with different images and different parameters in the reconstruction algorithm, we found that the success of this method depends on a number of different factors. The significant factors appear to be the accuracy of the zero-crossing points and the degree to which the zero-crossing points are spread out evenly throughout the picture. The required accuracy, usually a minimum of 12–14 digits in examples similar to Figs. 1 and 2, is likely to be the limiting factor in a number of potential applications in which the zero crossings cannot be measured accurately, either because of physical limitations or because of the presence of noise. The degree to which zero-crossing points are spread out evenly depends on the type of image as well as on the particular threshold used. For example, we note that the reconstruction of the image in Fig. 1 was more accurate than that of Fig. 2, despite the fact that the image of Fig. 2 contains fewer spectral components and the same number of equations were used, and that Fig. 1 contains more contours spread out throughout the picture.

To understand the effect of using different thresholds,

note that as the threshold is increased or decreased away from the mean, there will be fewer picture elements on one side of the threshold than the other and, furthermore, these picture elements will tend to be concentrated in small areas of the picture. This means that the threshold-crossing contours will be less evenly distributed throughout the picture, and the reconstruction process will be less stable. In both Fig. 1 and Fig. 2 the threshold chosen was somewhere near the mean value of the image. The mean value is not necessarily the best threshold to use, but the best threshold is likely to be fairly close to the mean in most images. While theoretically any threshold is adequate as long as it lies between the minimum and maximum values of the signal, i.e., as long as we have at least one threshold-crossing contour somewhere in the image, the threshold can significantly affect the stability of the reconstruction process. For most images, there is a range of thresholds for which the reconstruction works well, and outside this range significant errors occur that increase as the threshold varies further from this range.

Examples of reconstruction showing the effects of different thresholds are given in Fig. 3. This figure illustrates reconstruction using two different thresholds for the eye picture shown in Fig. 1. The images are on a scale of 0–1,

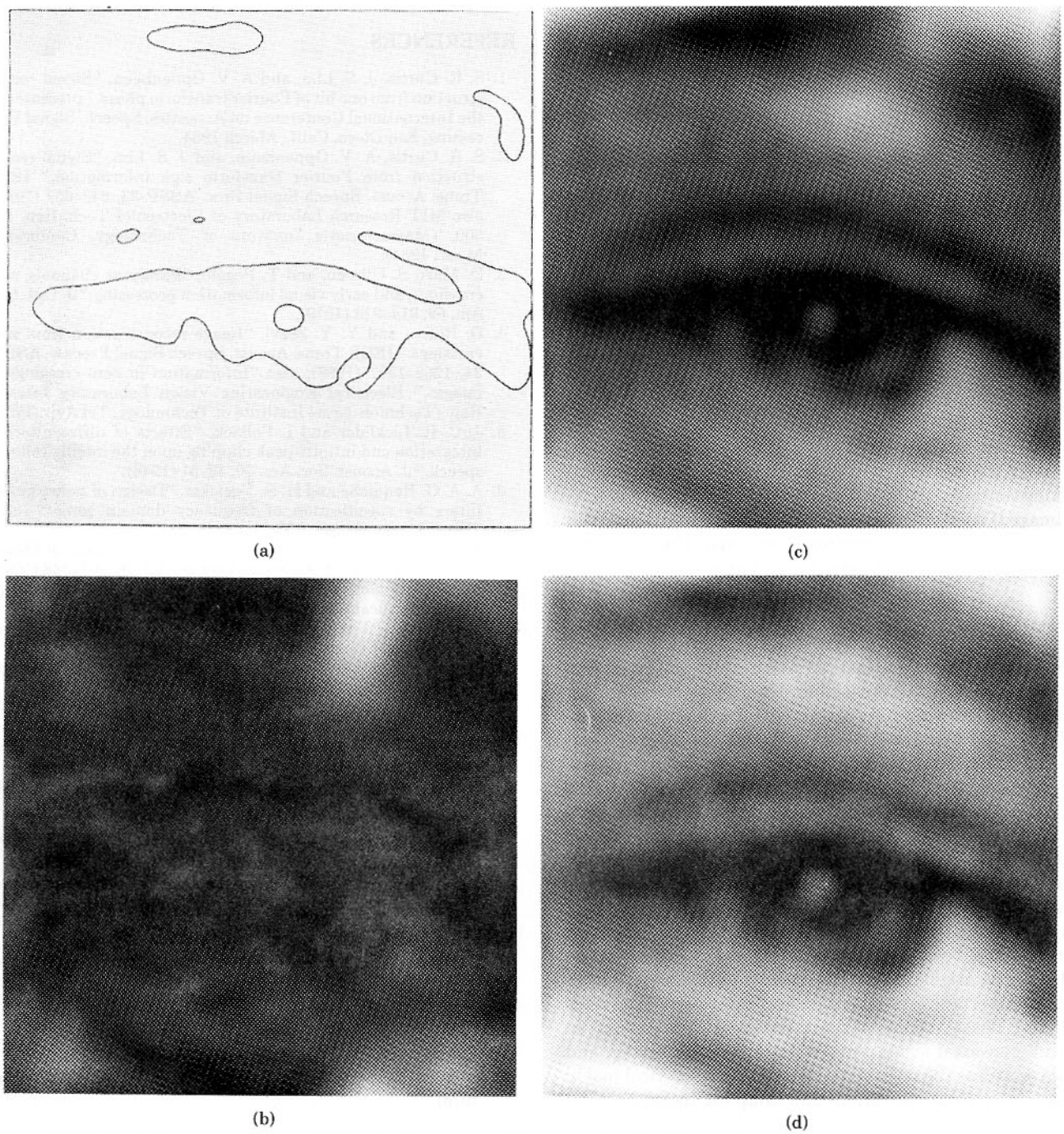


Fig. 4. Effect of additional equations: (a) threshold crossings, (b) recovered with 600 equations, (c) recovered with 750 equations, (d) recovered with 900 equations.

with mean values close to 0.5. Figure 3(a) shows the threshold-crossing contours obtained with a rather small threshold (0.27), such that most of the picture elements are brighter than this threshold and the threshold-crossing contours are concentrated in the center of the picture. Figure 3(b) shows the image recovered from these contours. Notice that there are significant errors in the corners of the image, areas that are farthest from any threshold-crossing contours. This type of error can be easily understood in terms of common experience with interpolation and extrapolation problems. Areas that are close to several zero-crossing contours are essentially found by interpolation, whereas areas far from zero-crossing contours are essentially found by extrapolation. Thus we would expect areas close to several zero-crossing contours to be recovered much more accurately than those far from any zero-crossing contours. Figures 3(c) and 3(d) show a larger threshold (0.64), where again we see distortions in areas far from the threshold-crossing contours. The threshold used for the reconstruction in Fig. 1 was approximately 0.5. For this image, the reconstruction is most successful in the range of thresholds between 0.30 and 0.62. The images shown in Fig. 3 illustrate the artifacts that occur at the edge of the range of acceptable thresholds. Farther from this range, the image bears little resemblance to the original.

One possible method of improving the accuracy of the reconstruction process is to increase the number of equations used. An example illustrating the effect of using additional equations is shown in Fig. 4. Figure 4(a) shows the threshold crossings of the eye image used in Fig. 3, with a threshold of 0.27. When 600 equations (454 unknowns) are used, the resulting image [Fig. 4(b)] has significant distortion near the corners, which are far from the threshold-crossing contours. When 750 equations are used, the resulting image [Fig. 4(c)] has improved, but the distortion is still noticeable. When 900 equations are used, the recovered image [Fig. 4(d)] appears very similar to the original.

5. CONCLUSIONS

In this paper we have presented new results on the unique specification of multidimensional signals with zero-crossing or threshold-crossing information. Our primary result established that two-dimensional, periodic, band-limited signals that are irreducible as polynomials are uniquely specified to within a scale factor by their zero-crossing contours. We also extended this result to permit finite-length signals and to permit crossings of an arbitrary threshold instead of zero crossings. In addition, we discussed extensions to signals with dimensions higher than two and to nonperiodic signals. Since previous results on unique specification of two-dimensional signals with zero crossings have required that the signal be bandpass or periodic or that a sine wave be added to the original signal, the results in this paper represent an important generalization and extension of previous results. These results suggest practical applications in multidimensional signal processing, image processing, and vision as well as the possibility for use as an analytical tool in areas such as communications and sampling theory.

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