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FREQUENCY WARPING IN THE DESIGN AND IMPLEMENTATION OF FIXED-POINT AUDIO EQUALIZERS

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ABSTRACT

This paper discusses the use of frequency warping for designing and implementing a class of discrete-time filters. This technique is particularly useful for audio filters because specifications are often given on a logarithmic frequency scale. It is shown that frequency warping allows a class of recursive filters to be designed using standard FIR techniques, and naturally leads to a structure for implementing the filters. The fixed-point behavior of this filter structure is analyzed and is shown to be relatively insensitive to coefficient quantization and round-off noise.

1. INTRODUCTION

In this paper, we apply the technique of frequency warping to the problem of designing filters for speech and audio applications. In these applications, logarithmically spaced frequency specifications often arise due to the natural frequency sensitivity of the human auditory system. As an example of a design problem in this context, consider the problem of designing a filter to approximate the magnitude specification shown in Figure 1. The desired filter characteristics are specified on a logarithmic frequency axis and are typical of the requirements placed on audio equalizers used to compensate for room modes and loudspeaker responses. When viewed on a linear frequency scale, the majority of the detail in the frequency response occurs below 5 kHz and this causes difficulty for typical filter design techniques that minimize error on a linear frequency scale. In particular, the dense low frequency details cause problems for the standard FIR filter design algorithms. Regardless of whether the "error" is actually measured in the least-square or Chebyshev sense, an FIR filter which fits the specifications shown in Figure 1 reasonably well requires at least several hundred filter taps. While it is possible to consider weighted least-square with an error weighting function which is heavily biased towards the low-frequency region, this approach tends to suffer from numerical problems caused by the narrow spacing of the low-frequency region. This eventually limits the order of the filter which can be designed. Likewise, the Parks-McClellan algorithm suffers numerical problems and fails to converge for the required filter lengths. In particular, our experience shows that the Parks-McClellan algorithms fails to converge for *arbitrary* (not piecewise constant) responses when the filter order is greater than 35, even with the use of double precision arithmetic in the design.

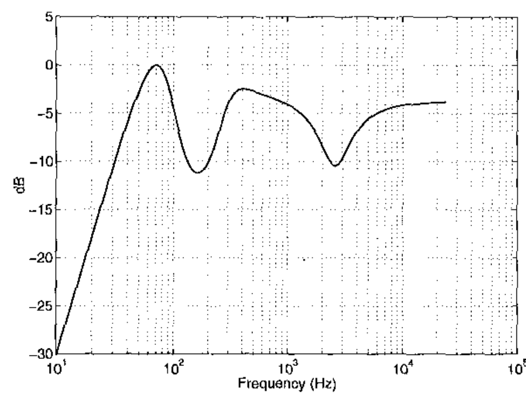

Figure 1: Example of a typical audio equalizer's frequency response, plotted on a logarithmic frequency scale.

An IIR filter would potentially offer a significant reduction in filter order. However, an IIR filter requires very high precision in both its design and actual operation. Furthermore, most IIR techniques do not directly address the design of a filter with an arbitrary magnitude response, and convergence to an acceptable solution is not guaranteed.

The approach taken in this paper is to apply a technique referred to as frequency warping. This general technique has previously been explored and developed in a number of contexts including filter design and used to approximate logarithmic frequency responses [1]-[7]. In this paper, we apply it specifically to the design of audio equalizers.

The allpass warping technique can be viewed as a hybrid of both FIR and IIR methodologies: it offers the flexibility of an FIR design algorithm, yet requires a filter order typical of an IIR filter. This increased robustness is achieved by recognizing that it is much simpler to design a filter to fit a desired specification $D(\omega)$ on a log-scale than on a linear scale. Thus by seeking a method of transforming the specification from $D(\omega)$ to $D(\Theta^{-1}(\omega))$, where $\Theta(\omega) \approx \log(\omega)$, the resulting filter order should be much smaller, and the design technique much more robust.

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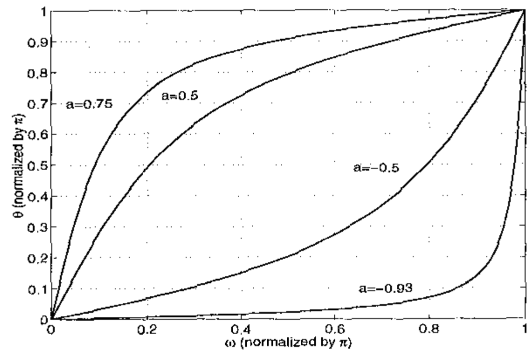

Figure 2: Relationship between the warped (ω) and the prototype (θ) frequency variables for warping factors (from left to right) $a = 0.75, 0.5, -0.5, -0.93$.

2. FILTER DESIGN USING FREQUENCY WARPING

Let $h[n]$ be an arbitrary real-valued impulse response with z-transform $H(z)$. A function $G(z)$ is an allpass-warped version of $H(z)$ if $G(z)$ can be obtained from $H(z)$ by replacing every delay with an allpass filter, i.e.,

$$G(z) \quad = \quad H(\Theta_a(z)) \tag{1}$$

where

$$\Theta_a^{-1}(z) = \frac{z^{-1} - a}{1 - az^{-1}} \text{ where } |a| < 1.$$

The subscript a is used to emphasize the dependence of the frequency response on the warping factor a.

Let $G(e^{j\omega})$ and $H(e^{j\theta})$ denote the Fourier transforms of $g[n]$ and $h[n]$ respectively. Note that different frequency variables are used for each transform. $G(e^{j\omega})$ and $H(e^{j\theta})$ are related by the substitution

$$\begin{aligned} G(e^{j\omega}) &= H(\Theta_a(e^{j\omega})) \\ &= H(e^{j\theta})\Big|_{\theta=\theta_a(\omega)} \end{aligned}$$

where the second equality follows from the fact that an allpass function maps the unit circle to itself. Thus, the phase response of the allpass function $\theta_a(\omega)$ describes the mapping between the two frequency variables. It then follows that

$$\theta_a(\omega) = \arctan \frac{(1-a^2)\sin\omega}{-2a + (1+a^2)\cos\omega}. \tag{2}$$

The parameter a in $\theta_a(z)$ is called the *warping factor*. It is a free parameter that controls the warping from $H(z)$ to $G(z)$. Examples of the function in Eq. (2) for a few values of a are plotted in Figure 2. The function $\theta_a(\omega)$ is one-to-one on the interval $[0, \pi]$ and its inverse is simply another allpass filter whose warping factor is $-a$. That is,

$$\theta_a^{-1}(\omega) = \theta_{-a}(\omega).$$

For $-1 < a < 0$, frequency warping stretches the low frequencies and compresses the high frequencies. Similarly, for $0 < a < 1$, the effect is the opposite — low frequencies are compressed and high frequencies stretched. This stretching and compressing is the key benefit of frequency warping and yields a substantial reduction in filter order.

A filter with a log-scale frequency specification can be warped with $-1 < a < 0$ to approximately linearize the frequency scale. A standard design technique is then applied to match this linear frequency response. Finally, the filter is warped again using $-a$ to return to the desired frequency scale.

3. IMPLEMENTATION

In this section, we show that frequency warping also leads to numerically robust filter structures, and analyze in detail the results of coefficient quantization and round-off noise.

Frequency warping is accomplished by the substitution of variables

$$z^{-1} \rightarrow \Theta_a^{-1}(z) = \frac{z^{-1} - a}{1 - az^{-1}} \text{ where } |a| < 1 \tag{3}$$

into the system function of a prototype filter $G(z)$. This substitution can also be performed directly on the system *structure* as shown in Figure 3. Specifically, every delay in the system structure is replaced by an allpass filter.

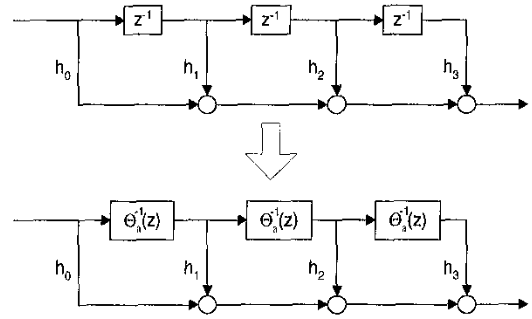

Figure 3: Direct substitution of delays with allpass filters.

Direct replacement of each delay by an allpass transfer function limits our consideration to structures obtained from FIR prototypes. This is because direct replacement in IIR filters leads to delay-free loops. This problem has been recognized and discussed in [8] and [9]. Therefore, while the paper has thus far considered frequency warping applied to arbitrary filters, both IIR and FIR, the rest of this section is devoted to warped FIR filters.

3.1. Fixed-Point Implementation

This section presents an analysis of the warped FIR filter in the presence of fixed-point arithmetic. It is shown that by selecting a suitable allpass filter topology, the warped FIR structure yields a numerically robust scheme for implementing audio equalizers.

We assume that a modern digital signal processor is the target platform for the warped FIR filter. A DSP contains fixed bit-width memory and registers, and typically has a large bit-width accumulator for storing intermediate results. Quantization and truncation generally occur only when data is transferred from the accumulator back into a register or memory location, or output from the system.

This quantization model differs from the traditional method of introducing a quantization error following every multiplication [5]. In terms of analyzing a signal flow graph, quantization occurs only when:

1. A multiplication is not uniquely and immediately followed by an addition, or
2. An addition is not uniquely and immediately followed by another addition.

We further assume that B-bit signed fractional arithmetic is used. Under this assumption, there are 2^B allowable values ranging from -1 to $1-\Delta$ where $\Delta = 2^{-B+1}$ represents a change in the least significant bit.

The errors resulting from fixed-point arithmetic can be placed into 3 main categories: coefficient quantization, signal overflow, and numerical noise. We analyze each type of error separately.

The warping factor a and the FIR coefficients $h[n]$ are typically designed using double precision arithmetic. These must then be quantized to the nearest representable value. Since there is some flexibility in the choice of warping factor, we assume that the warping factor is selected *a priori* from the set of allowable numerical values. Thus, the warping factor will not undergo any quantization. Once the warping factor is selected, the FIR coefficients $h[n]$ are designed and then quantized. It is well-known that the frequency response of FIR filters is relatively insensitive to the effects of coefficient quantization [5], and by extension, warped FIR filters are relatively insensitive as well.

Signal overflow occurs when the output of the filter, or a result stored in an intermediate delay element, exceeds the allowable range of $-1 \leq x \leq (1-\Delta)$. Although it is difficult to fully characterize the specific input conditions under which an overflow occurs, it is possible to develop some useful heuristics. A good starting point is to assume that the gain from the input of the system to all intermediate delay elements is less than or equal to 1 for all frequencies. If this condition is satisfied, then overflows will be completely avoided for sine wave inputs (in the steady state). If the gain exceeds 1, then the input must be scaled down accordingly, and this reduces the overall signal-to-noise ratio (SNR) of the filter. By applying this heuristic to the four standard topologies for allpass filters, it can be shown that Direct Form I is best in that it requires no gain reduction of the input signal. Thus, it is least prone to overflow and leads to the highest SNR.

The main apparent drawback of the Direct Form I structure is that it requires an additional delay element between every allpass stage. However, by merging the delay elements of neighboring stages a cascade of N allpass filters can be implemented using only N+1 delay elements — one more than the canonical version. The final filter topology, including all quantization errors, is shown in Figure 4.

We make three assumptions regarding the quantization noise, and these parallel the presentation in [5]:

1. Each quantization noise $e_i[n]$ is a wide-sense stationary white noise process.
2. Each noise source has a uniform distribution over the range $\pm\frac{1}{2}\Delta$. Thus the variance of each noise source is $\sigma_e^2 = \frac{2^{-2B+2}}{12}$.
3. Each quantization noise source is uncorrelated with the input to the quantizer, all other quantization noise sources, and the input to the system.

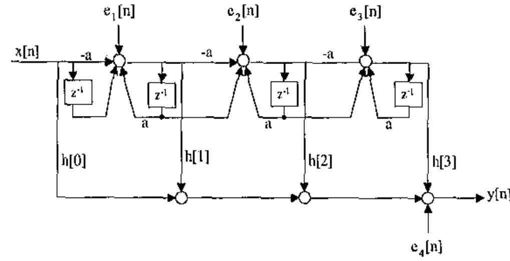

Figure 4: Warped FIR structure with additive noise shown. $e_1[n]$, $e_2[n]$, $e_3[n]$ are due to the allpasses, while $e_4[n]$ is due to the weighted sum with FIR coefficients.

Since we assume that all of the noise sources are uncorrelated, the overall SNR of the filter is derived by summing all of the noise contributions at the output of the filter. Define $H_i(z)$ as the partial transfer function $H_i(z) \triangleq \sum_{k=i}^{M-1} h[k]z^{-(k-i)}$. Then the transfer function from the i^{th} noise source to the output would be $\frac{H_i(\Theta_a(z))}{1-az^{-1}}$. The noise spectrum at the filter's output can be expressed as

$$\sigma_e^2 \sum_{i=1}^{M-1} \left| \frac{H_i(\Theta_a(e^{j\omega}))}{1-ae^{-j\omega}} \right|^2 + \sigma_e^2$$
$$= \frac{\sigma_e^2}{|1-ae^{-j\omega}|^2} \left(\sum_{i=1}^{M-1} \left| H_i(\Theta_a(e^{j\omega})) \right|^2 \right) + \sigma_e^2. \quad (4)$$

It is difficult to draw general conclusions about the output noise spectrum because it is a function of the specific filter coefficients used. However, most of the terms in (4) have a common denominator of $1/\left|1-ae^{-j\omega}\right|^2$. The common term represents the noise shaping due to the pole in the allpass filter, and we can think of the output noise as being shaped by this first order low pass filter. In practice, we have found that the output noise spectrum is heavily influenced by this low pass characteristic. This is due to the choice of a which is typically close to 1. This is ideal for audio applications due to the decreased sensitivity of the human ear to low frequencies. Also, audio generally has more energy at low frequencies, and this tends to further mask the quantization noise.

3.2. Design Example

This section contains a complete example including design, implementation, and analysis of the resulting filter to the specification in Figure 1. The procedure consists of warping the specification with parameter a, designing an FIR filter, and implementing the warped-FIR filter with parameter $-a$.

The optimal warping factor is defined as the value of a which minimizes the length of the FIR filter. We determined the optimal warping factor by iteratively repeating the design procedure and searching for the smallest filter length which still met the specifications. The optimal value of a is usually close to the warping parameter which best linearizes the original log-frequency specifications, and we used this as a starting point. For the filter in Figure 1, the optimal warping factor was found to be $a = 0.963$.

Furthermore, additional saving in filter length may be achieved by designing it to be minimum-phase. We used a standard least-

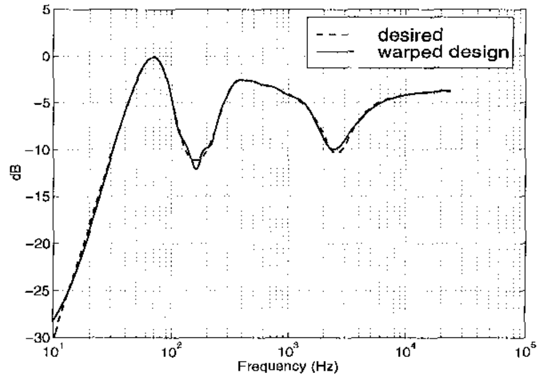

Figure 5: Comparison of the desired (solid) and resulting (dashed) frequency responses.

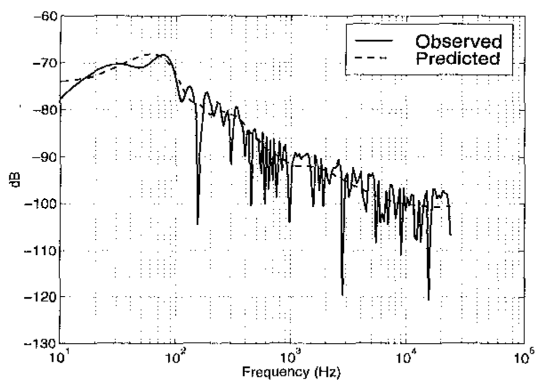

Figure 6: Comparison of measured (solid) and predicted (dashed) quantization noise power spectral densities when 16-bit arithmetic is used. Length FIR prototype=21, $a = 0.963$.

square design to obtain a linear-phase filter, then performed a spectral factorization on it. This yielded a 21-point prototype filter. The resulting frequency response is shown in Figure 5. As can be seen, the warped filter is able to match the frequency response even at relatively low order.

Figure 6 illustrates the noise at the output of the system for the case of 16-bit arithmetic. The solid curve represents the quantization noise measured at the output of the system (defined as the difference between the output of a 16-bit filter and a double precision filter, using white noise as the input). The dashed curve represents the noise predicted by Equation 4. Note that there is good agreement between the theoretically predicted and actually observed noise. Also, the overall shape of the noise spectrum is dominated by the lowpass term $1/\mid 1 - ae^{-j\omega} \mid$ as mentioned earlier.

4. CONCLUSION

In this paper, the theory and some practical aspects of frequency warping based on allpass filters have been discussed. Allpass warping allows one to derive a filter from a prototype by replacing every delay with an allpass filter. One of the main benefits of frequency warping is its ability to yield a numerically robust solution for difficult filter design problems. This is especially true for filters whose specifications are highly nonuniform such as audio filters specified on a log-frequency scale. We also presented a structure for implementing filters using fixed-point arithmetic. This was based upon a direct substitution of allpass filters directly into the system structure of a filter. Simulation results confirm that the quantization noise output is relatively low and has a low-pass characteristic.

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