PROBABILISTIC COMPLEXITY ANALYSIS OF INCREMENTAL DFT ALGORITHMS

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ABSTRACT

We present a probabilistic complexity analysis of a class of multi-stage algorithms for computing successive approximations to the DFT. While the quality of the approximate spectra obtained after any stage of these algorithms can be readily quantified in terms of commonly used input-independent metrics of spectral quality, each stage's arithmetic complexity is dependent on the nature of the input signal. Modeling the input signal as a stationary Gaussian-distributed random process, we obtain estimates of the distribution of the number of arithmetic operations required to complete any algorithm stage. This enables the derivation of important design information such as the probability with which a desired quality of approximation is achieved within a given arithmetic bound. Our results are verified using a Monte Carlo analysis.

1. INTRODUCTION

We have recently introduced a class of algorithms for computing approximations to the DFT which we refer to as DFT incremental-refinement (DFT-IR) algorithms [1]. The term *incremental refinement* denotes the fact that each of the algorithms consists of multiple stages, each of which improves upon the DFT approximation produced by the previous stage. The incremental refinement property gives the DFT-IR advantages over more commonly used approaches to DFT approximation, such as FFT pruning, for real-time systems with unpredictable or time-varying restrictions on the availability of computing resources [2] [3]. For such applications, the use of incremental refinement algorithms enables the tradeoff obtained between approximation quality and the amount of computation performed to be easily modified by varying the number of stages performed before algorithm termination.

The selection of an appropriate DFT-IR algorithm for a given application requires that the relationship between approximation quality and arithmetic cost be adequately quantifiable. The quality of the DFT approximation obtained after each stage of a DFT-IR algorithm may be characterized in terms of commonly used input-independent metrics for spectral quality: SNR, frequency resolution, and frequency coverage. The arithmetic complexity of each stage, however, is dependent upon the characteristics of the input signal.

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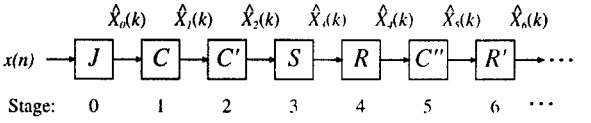

Figure 1: Block diagram depiction of the first seven stages of a DFT-IR algorithm.

We have derived expressions for the probability distribution associated with the number of arithmetic operations needed to complete any DFT-IR algorithm stage based on the assumption of a stationary Gaussian-distributed random input signal [4]. Exact evaluation of these expressions, however, is computationally intractable. In this paper, we present a computationally tractable technique for obtaining estimates of the distribution. This enables the derivation of important design information, such as the probability with which a desired quality of approximation is achieved within a given arithmetic bound. Our results are verified using a Monte Carlo analysis.

2. DFT-IR ALGORITHMS

Every DFT-IR algorithm can be viewed as a cascade of stages, each of which takes a DFT approximation $\hat{X}_{i-1}(k)$ and produces an improved approximation $\hat{X}_i(k)$. This structure is illustrated using a block-diagram format in Fig. 1. The refinement process is "jump-started" with the computation of an initial approximation $\hat{X}_0(k)$ as indicated by the block of type J in the figure. All subsequent stages perform one of three different updates, each of which improves the previous approximation by a fixed amount in either SNR, indicated by blocks of type S, in frequency resolution, indicated by blocks of type R, or in frequency coverage, indicated by blocks of type C. The number of arithmetic operations performed in each block is dependent upon the values of the input data.

Every unique sequence of updates corresponds to a different DFT-IR algorithm. We represent each of these algorithms using a set of control parameters, s_i, r_i, and c_i. For each i, the control parameter values essentially represent the number of updates of the corresponding type (S, R, or C) that are present up to and including the ith stage. The "jump-start" stage is counted as an update of all three

types.

The DFT-IR algorithms perform each of their updates without multiplications using a distributed arithmetic technique based on the summation of pre-computed vectors [5]. With this technique, the total arithmetic cost through stage i, which we denote by κ_i, is [1]

$$\kappa_i = s_i r_i + 2\gamma_i c_i \quad \text{real additions,} \tag{1}$$

where

$$\gamma_i = \sum_{q=1}^{s_i} \sum_{n=0}^{r_i-1} |g_q(n)|. \tag{2}$$

The quantity γ_i represents the signal-dependent contribution to the arithmetic cost of completing the ith stage of processing and is equal to the number of non-zero elements in a restricted portion of the backward difference of the quantized input signal. We therefore refer to it as the *non-zero count* for stage i. The function $g_q(n)$ represents the first circular backward difference of $x_q(n)$, the qth bit vector[1] of the two's complement binary representation of the signal under analysis.

3. PROBABILISTIC COMPLEXITY ANALYSIS

To obtain a probability distribution for κ_i, the total arithmetic cost associated with completing stage i of a DFT-IR algorithm, we suppose that the signal under analysis is obtained via amplitude quantization of a continuous-valued discrete-time Gaussian-distributed random process, denoted $\tilde{x}(n)$, with known autocorrelation. Expressions for the exact distribution of κ_i have been derived under these conditions, however the amount of computation required for their evaluation grows combinatorially with s_i and r_i [4].

By adopting additional simplifying assumptions about the probabilistic behavior of $g_q(n)$, we have derived expressions that allow estimates of the distribution of κ_i to be obtained using a number of computations that grows in a polynomial fashion with s_i and r_i. These assumptions are:

(i) It is assumed that the values of $g_q(0)$ make insignificant contributions to the non-zero count.

(ii) The interactions[2] between elements of $g_q(n)$ of third and higher order are assumed to be negligible.

(iii) Any sample of $g_j(n)$ is assumed to not be significantly correlated with any sample of $g_k(n)$ when $j \neq k$.

Each of these assumptions in some way facilitates a tractable analysis of the distribution of the arithmetic cost while introducing some error into its results. Assumption (i) is used because while $g_q(n_0) = x_q(n_0) - x_q(n_0 - 1)$ for $n_0 > 0$, the sample $g_q(0)$ depends upon the difference $x_q(0) - x_q(N-1)$. Our probabilistic analysis is simplified by ignoring this exception. This is reasonable because generally $g_q(0)$ is only one of many samples in $g_q(n)$ which contribute to the non-zero count. Our second assumption is based on the observation that interactions between elements of $g_q(n)$ diminish with increasing order. Thus, restricting our consideration to the first and second order moments can be expected to capture the most important characteristics of the distribution. The implications of this approximation have been investigated in detail previously. Bounds on its error are known and it has been shown experimentally to be reasonable [6]. While we are not aware of any formal analysis to support assumption (iii), our empirical results (as reported in Section 4) indicate that this assumption did not excessively degrade the accuracy of distribution obtained.

Since κ_i is deterministically related to the non-zero count, γ_i, we focus on the derivation of the distribution $p_{\gamma_i}(k) \triangleq \text{Prob}\{\gamma_i = k\}$. Under our stated assumptions, this distribution can be expressed as:

$$p_{\gamma_i}(k) = p_{\gamma_i^1}(k) * p_{\gamma_i^2}(k) * \cdots * p_{\gamma_i^{s_i}}(k), \tag{3}$$

where

$$\begin{aligned} p_{\gamma_i^q}(k) = {} & R_{zz}(q,0)^k (1 - R_{zz}(q,0))^{r_i - k} \\ & \times \left[\binom{r_i}{k} + \sum_{n=1}^{r_i-1} (r_i - n) \frac{R_{zz}(q,n) - R_{zz}(q,0)^2}{R_{zz}(q,0)(1 - R_{zz}(q,0))} \right. \\ & \times \sum_{w=\max(0,k-r_i+2)}^{\min(2,k)} \binom{2}{w} \binom{r_i - 2}{k - w} \\ & \times \left(\frac{-R_{zz}(q,0)}{\sqrt{R_{zz}(q,0)(1 - R_{zz}(q,0))}} \right)^{2-w} \\ & \left. \times \left(\frac{1 - R_{zz}(q,0)}{\sqrt{R_{zz}(q,0)(1 - R_{zz}(q,0))}} \right)^{w} \right]. \end{aligned} \tag{4}$$

We use γ_i^q to denote the non-zero count over $0 \leq n \leq r_i - 1$ of the backward difference vector $g_q(n)$ and $p_{\gamma_i^q}(k)$ to denote its distribution. The quantity $R_{zz}(q,n)$ represents the the autocorrelation of the random processes $z(q,n) = |g_q(n)|$.[3] $R_{zz}(q,n)$ is related to the distribution of the random process $\tilde{x}(n)$ by:

$$R_{zz}(q,n) = \begin{cases} \displaystyle\iint_{\substack{(x,y): \\ \xi_q(x) \neq \xi_q(y)}} p_{\tilde{x}(n-1)\tilde{x}(n)}(x,y)\, dx\, dy, & n = 0, \\ \displaystyle\iiint_{\substack{(x,y,z): \\ \xi_q(x) \neq \xi_q(y) \neq \xi_q(z)}} p_{\tilde{x}(n-1)\tilde{x}(n)\tilde{x}(n+1)}(x,y,z)\, dx\, dy\, dz, & n = 1 \vee n = N-1, \\ \displaystyle\iiiint_{\substack{(w,x,y,z): \\ \xi_q(w) \neq \xi_q(x) \wedge \\ \xi_q(y) \neq \xi_q(z)}} p_{\tilde{x}(-1)\tilde{x}(0)\tilde{x}(n-1)\tilde{x}(n)}(w,x,y,z)\, dw\, dx\, dy\, dz, & 2 \leq n \leq N-2. \end{cases} \tag{5}$$

The function $\xi_q(x)$ represents the input/output relation for the qth bit of a two's complement binary quantizer and N is the length of the DFT being approximated. Determination

[1] For a Q-bit signal representation, we use $q = 1$ to denote the MSB and $q = Q$ for the LSB.

[2] We say that random variables interact when the expected value of their product is not equal to the product of their expected values.

[3] We note that $z(q,n)$ is made stationary in time by assumption (i) and independent across bit levels by assumption (iii).

of $R_{zz}(q,n)$ requires the evaluation of the multivariate normal integral. There are only a few special cases for which closed form solutions exist, so for this task we must rely on numerical methods.

Careful inspection of Eq. (4) reveals that the distribution of γ_i^q has a similar form to that of the binomial distribution. In fact, when there are no second order interactions between distinct elements of $g_q(n)$, a binomial distribution with parameter $R_{zz}(q,0)$ is obtained. This equation can be viewed as providing the corrections to the binomial distribution that are required due to correlation between the elements of $g_q(n)$. The derivation of Eqs. (3)-(5) is based on a representation of correlated binomial random processes in terms of its moments and is detailed in another publication [4].

4. EXPERIMENTAL VERIFICATION

We have performed a series of experiments to evaluate the accuracy of Eqs. (1)-(5) at predicting the arithmetic complexity of DFT-IR algorithms applied to stationary Gaussian-distributed inputs. These experiments indicate that our theoretical results provide reasonable estimates of DFT-IR algorithms' performance for this important class of signals.

The signal statistics employed in this evaluation correspond to a reported long-term average for male speech [7]. The variance of the input process was normalized to 0.25, representing a well-scaled input to a quantizer with amplitude range $[-1,1]$. Signal quantization to 16 bits was applied to obtain a signal suitable for DFT-IR analysis and a 32 kHz sampling rate was employed. We assumed the signal under analysis to be contained in a 128-point frame that is to undergo a 256-point DFT.

The control parameters and output quality associated with selected stages from two DFT-IR algorithms are listed in the first eight columns Table 1. Fig. 2(a)-(b) shows the cumulative distribution of the arithmetic cost associated with these algorithms stages (i.e. Prob $\{\kappa_i \leq x\}$) as predicted by Eqs. (1)-(5). The cumulative distributions of the computational complexity of these same algorithm stages were determined empirically by monitoring their arithmetic cost in 50,000 Monte Carlo trials. These experimentally derived distributions are shown in Fig. 2(c)-(d).

These cumulative distributions represent the probability with which the computational cost of completing a particular algorithm stage is less than or equal to a given value. They can therefore be considered to provide the *probability of completion* for that stage within any specified arithmetic cost. Probability of completion is an important figure of merit for a variety of resource-constrained design contexts [4]. The ninth and tenth columns of Table 1 present the theoretically predicted and experimentally obtained probabilities with which each of the selected stages are completed using not more than 1000 arithmetic operations. These values represent the intersections of the distributions of Fig. 2 with the vertical grid-line at an arithmetic cost of 1000 operations.

Our experimental results indicate that reasonable estimates of the probabilities of completion are obtained through application of the theoretical analysis of Section 3. This tends to validate the approximations that were made in our derivations. The agreement between the theoretically determined probabilities and those that were obtained experimentally can be seen to diminish as the SNR of the approximation is increased. This effect can be attributed to our assumption of independence between bit levels.

5. CONCLUSION

We have presented a technique for the probabilistic complexity analysis of the DFT-IR family of algorithms. Using stationary Gaussian signal models, our results enable the calculation of reasonable estimates of the probability distribution associated with the arithmetic cost of applying a DFT-IR algorithm to generate a DFT approximation of a desired spectral quality. The utility of these results has been illustrated in the context of deriving the probability with which selected DFT-IR algorithm stages are completed using not more than a specified number of arithmetic operations.

The distributions derived here are useful for addressing a variety of other design problems, many of which have yet to be explored. For example, for applications with a fixed arithmetic bound on the number of operations expended in the execution of a DFT-IR algorithm, the probability distribution over spectral approximation quality can be easily obtained. Such an analysis could be used as the basis for selecting the DFT-IR algorithm whose sequence of update stages is best suited for a particular application based on time-dependent refinement criteria. Another potential application of these results, currently under investigation by the authors, is for the evaluation of various system-level strategies for run-time allocation of computing resources to successive DFT frames in a real-time spectrum analyzer.

6. REFERENCES

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Algorithm	*Stage*	*Control Parameters*			*Output Quality*			*Probability of Completion (Theoretical)*	*Probability of Completion (Measured)*
	i	s_i	r_i	c_i	*SNR*	*Resolution*	*Coverage*		
1	70	1	8	64	6	8	$\pi/2$	0.999	0.999
	78	1	16	64	6	16	$\pi/2$	0.967	0.956
	94	1	32	64	6	32	$\pi/2$	0.635	0.628
	126	1	64	64	6	64	$\pi/2$	0.154	0.137
2	70	1	8	64	6	8	$\pi/2$	0.999	0.999
	71	2	8	64	12	8	$\pi/2$	0.894	0.841
	72	3	8	64	18	8	$\pi/2$	0.353	0.421
	73	4	8	64	24	8	$\pi/2$	0.042	0.095

Table 1: Description and analysis of four stages taken from each of two different 256-point DFT-IR algorithms. SNR is given in dB, frequency resolution is the maximum number of resolvable frequency components, and frequency coverage is in radians. Probabilities resulting from the theoretical and Monte Carlo analyses are given for the completion of each stage within 1000 arithmetic operations. The input signal is presumed to have the long-term average spectrum of male speech.

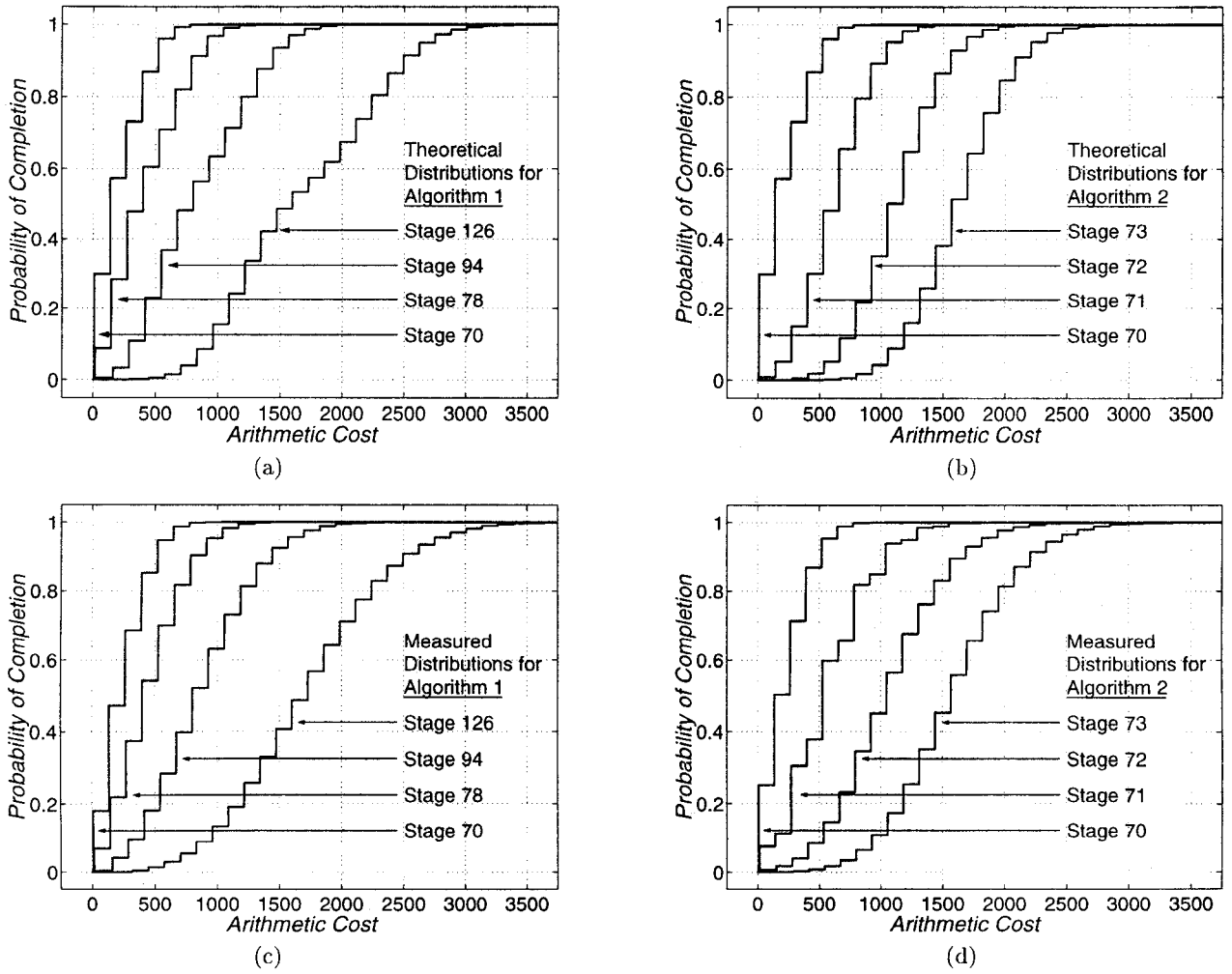

Figure 2: Cumulative probability distributions of the total arithmetic cost associated with completing the DFT-IR algorithm stages given in Table 1. (a) Theoretically predicted distributions for Algorithm 1, Stages 70, 78, 94, 126. (b) Theoretically predicted distributions for Algorithm 2, Stages 70, 71, 72, 73. (c) Experimentally determined distributions for Algorithm 1, Stages 70, 78, 94, 126. (d) Experimentally determined distributions for Algorithm 2, Stages 70, 71, 72, 73.