PHASE-ONLY SIGNAL RECONSTRUCTION*

Monson H. Hayes, Jae S. Lim, Alan V. Oppenheim

M.I.T. Lincoln Laboratory

Lexington, MA.

ABSTRACT

In this paper, we develop a set of conditions under which a sequence is uniquely specified by the phase or samples of the phase of its Fourier transform. These conditions are applicable to mixed-phase one-dimensional and multi-dimensional sequences. Under the specified conditions, we also present several algorithms which may be used to reconstruct a sequence from its phase.

INTRODUCTION

In a variety of practical applications, it would be desirable to be able to reconstruct a sequence from the phase of its Fourier transform. For example, in blind deconvolution a desired signal is to be recovered from an observation which is the convolution of the desired signal with some unknown signal. Since little is usually known about either the desired signal or the distorting signal, deconvolution of the two signals is generally a very difficult problem. However, consider the special case in which the distorting signal is known to have a phase which is identically zero. Such a situation occurs, at least approximately, in long-term exposure to atmospheric turbulence or when images are blurred by severely defocused lenses with circular aperture stops (1). In this case, except for phase reversals, the phase of the observed signal is identical to the phase of the original signal. Therefore, if phase-only signal reconstruction were possible, the deconvolution could be performed exactly. In a related problem, phase-only signal reconstruction might also be used in the estimation of the frequency response of a linear time-invariant system if, for example, the symmetry of an input to the system could be controlled. As another example, in a Fourier transform coding system, both the magnitude and phase are usually coded and transmitted (2). However, for signals which can be recovered from only the phase, unnecessary redundancy is inherent in the coder. Therefore, for these signals it may be possible to realize a significant bit-rate reduction by simply coding the phase and then reconstructing the sequence at the receiver from the coded phase. Thus, it is of considerable importance to determine conditions under which a sequence is uniquely specified by its phase and to develop techniques to reconstruct a sequence from it phase under such conditions.

*This work was sponsored by the Dept. of the Air Force

In general, a sequence is not uniquely defined by its phase, as is illustrated by the observation that a sequence convolved with any zero phase sequence will produce another sequence with the same phase. Thus, without some assumptions about the sequence, the phase may, at best, uniquely specify a sequence only to within an arbitrary zero-phase factor. However, if some additional knowledge is available about a sequence, then under certain conditions the sequence may be uniquely defined by its phase. For example, if it is known that all the poles and zeroes of a sequence are within the unit circle, then the sequence is minimum phase and thus uniquely defined to within a scaling factor by its phase (3).

Recently (4), we have developed new conditions under which a sequence is uniquely defined by its phase. These conditions, which are applicable to both one-dimensional (1-D) and multi-dimensional (n-D) sequences, are potentially applicable to a variety of practical problems. Furthermore, we have developed several numerical algorithms to reconstruct a sequence from its phase under the specified conditions. The purpose of this paper is to review some of these recent results. In the next section, we summarize the theoretical results pertaining to the phase-only signal reconstruction problem. In the final section, we describe two numerical algorithms which have been developed to perform this reconstruction.

THEORETICAL RESULTS

As mentioned in the introduction, a sequence is not uniquely defined by its phase without some additional knowledge about the sequence. However, it has recently been shown that under relatively loose constraints, a finite length sequence is recoverable from its phase. More specifically, a phase-only reconstruction theorem has been developed (4) which states:

> Theorem 1: A 1-D sequence which is finite in length and has a z-transform with no zeroes on the unit circle and no zeroes in conjugate reciprocal pairs in uniquely specified to within a scaling factor by the phase of its Fourier transform (or by the tangent of its phase).

CH1559-4/80/0000-0437$00.75

The condition which excludes zeroes from the unit circle is made only for convenience. The condition which excludes zeroes in conjugate reciprocal pairs, however, is necessary to eliminate the possible ambiguity due to zero-phase components. This theorem is also applicable to all-pole sequences since the convolutional inverses of these sequences are finite in length.

Although Theorem 1 is formally stated for 1-D sequences, an extension to n-D sequences has been accomplished by using the projection-slice theorem. This theorem establishes the result that an n-D sequence having a rational z-transform may be mapped into a 1-D sequence (projection) by means of an invertible transformation (5). This transformation has the important property that the phase of the projection is uniquely defined by the phase of the n-D sequence. Specifically, the phase of the projection is equal to a slice of the phase of the n-D sequence. Consequently, the multi-dimensional phase-only problem can be mapped into a one-dimensional phase-only problem and the phase-only reconstruction theorem for 1-D sequences may be used.

The approach of transforming n-D sequences into 1-D projections provides at least a partial solution to the multi-dimensional phase-only problem. However, this approach circumvents the fundamental issues involved in multi-dimensional phase-only signal reconstruction. For example, this approach imposes constraints on a projection of an n-D sequence rather than directly on the n-D sequence. In addition, although it may not be possible to perform a phase-only reconstruction of an n-D sequence from a particular projection, this does not preclude the possibility that there exists another projection for which the reconstruction is possible. Therefore, with this approach it is difficult to determine which multi-dimensional sequences may be reconstructed from their phase. Recently, however, an extension of Theorem 1 to n-D sequences has been developed. This theorem states:

> Theorem 2: An n-D sequence which is finite in extent is uniquely specified to within a scale factor by its phase if its n-dimensional z-transform, $X(\underline{z})$, cannot be factored as
>
> $$X(\underline{z}) = F(\underline{z})\ G(\underline{z})$$
>
> where $F(\underline{z})$ and $G(\underline{z})$ are non-trival polynomials of finite order in $\underline{z}$ and $\underline{z}^{-1}$ with $F(\underline{z})=F(\underline{z}^{-1})$.

Clearly, Theorem 1 is a special case of this theorem. However, because of the absence of a Fundamental Theorem of Algebra for polynomials in more than one variable the proof of the general n-dimensional theorem is more abstract than that required in the one-dimensional case (the detailed proof will be published separatelv).

Although the phase-only reconstruction theorems specify a set of conditions under which a sequence is uniquely specified to within a scale factor by its phase, it is assumed that the phase is known for all frequencies. Since any practical algorithm for reconstructing a sequence from its phase will base the reconstruction on only a finite set of samples of the phase, the 1-D theorem has been extended to consider the uniqueness of a sequence based only on samples of its phase. Specifically,

> Theorem 3: A sequence which is known to be zero outside the interval $0 \leq n \leq (N-1)$ is uniquely specified to within a scale factor by $(N-1)$ distinct samples of its phase (or tangent of its phase) in the interval $0<\omega<\pi$ if it has a z-transform with no zeroes on the the unit circle or in conjugate reciprocal pairs.

This theorem forms the basis for demonstrating the existence and uniqueness of solutions to the signal reconstruction algorithms which are described in the next section. Unfortunately, however, Theorem 3 can not be directly extended to n-D sequences. In other words, an n-D sequence which is uniquely defined by its phase for all frequencies under the conditions of Theorem 2 is not, in general, uniquely defined by an arbitrary finite set of phase samples in the open region $\underline{0} < \underline{\omega} < \underline{\pi}$. The existence of such a set and a method for finding it is currently being investigated.

ALGORITHMS

Consider a finite duration sequence $x[n]$, which is zero outside the interval $0 \leq n \leq (N-1)$ and has no zeroes on the unit circle or in conjugate reciprocal pairs. From Theorem 3, we know that $x[n]$ is uniquely specified to within a scale factor by $(N-1)$ distinct samples of its phase in the interval $0 < \omega < \pi$. In this section, we describe two numerical algorithms which have been developed to recover $x[n]$ from samples of its phase. The first algorithm is an iterative procedure for which the total squared error is non-increasing with each iteration. The second algorithm is a closed form solution which involves solving a set of linear equations.

Iterative Algorithm

The mathematical problem to be solved in the phase-only signal reconstruction problem defined above is to find a Fourier transform pair which is consistent with the known constraints: The frequency domain constraint that the sequence has the appropriate phase samples and the time domain constraint that the sequence is zero outside the interval $0 \leq n \leq (N-1)$. An interative approach to finding $x[n]$ is a modified version of the Gerchberg-Saxton algorithm (6) which is a special case of the more general error-reduction approach proposed by Fienup (7). This method involves repeated transformation between the time and frequency domains with the appropriate constraints imposed in each domain. Thus, at the kth iteration, the current estimate of the sequence is Fourier transformed and the resulting phase is replaced with the correct phase. Inverse Fourier transforming, the (k+1)st estimate is formed by setting the points outside the interval $0 \leq n \leq (N-1)$ equal to zero. An illustration of this iterative algorithm using the Discrete Fourier Transform (DFT) and Inverse Discrete Fourier Transform (IDFT) in shown in Figure 1.

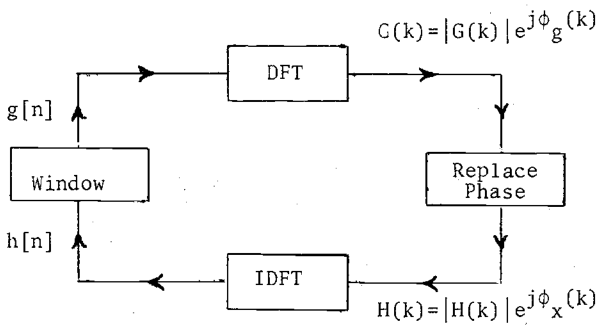

Figure 1: Iterative Algorithm

Since (N-1) distinct phase samples in the interval $0 < \omega < \pi$ are obtained when a DFT of length $M \geq 2N$ is used in the iteration, a unique solution to the phase-only problem exists in this case. However, the conditions under which the convergence of this iterative procedure is guaranteed are not presently known. In spite of this, it has been shown (4) that the total squared error between the kth estimate and the sequence x[n] is non-increasing with each iteration. Furthermore, it has also been shown that if the iteration converges and if $x[0] \neq 0$, then it converges to a scaled version of x[n].

In the examples which we have considered so far, the iteration has always converged to a scaled version of x[n] when $M > 2N$ and $x[0] \neq 0$. An example is shown in Table 1 for a mixed phase sequence of length 8 using a DFT of length 16. The initial estimate of x[n] was formed by using the correct phase samples and a DFT magnitude equal to a constant. In addition, at each iteration the sequence estimate was appropriately scaled so that the value at n=0 was equal to x[0]. As illustrated by this example, the number of iterations required to reach a convergent solution is, in general, very large. It is possible, however, to increase the rate of convergence. For example, using a "first-order adaptive acceleration technique" (8), the number of iterations required to achieve a given squared error is significantly reduced as shown in Table 1.

Closed Form Solution

A closed form solution to the phase-only signal reconstruction problem has been developed which involves finding the solution to a set of linear equations. The equations are derived from the definition of the phase function. Specifically, with $\phi_x(\omega)$ representing the phase of the sequence, x[n], it is easy to show (4) that

$$\sum_{n=1}^{N-1} x[n] \sin[\, \phi_x(\omega) + n\omega \,] = -x[0] \sin \phi_x(\omega) \tag{1}$$

When sampled at (N-1) distinct frequencies in the interval $0 < \omega < \pi$, this equation can be written in matrix form as

$$S\, \underline{y} = \underline{b} \tag{2}$$

where $\underline{y}$ represents the vector of elements of x[n] excluding the first element of the sequence, x[0]. It has been shown (4) that a unique solution to eq. (2) exists if $x[0] \neq 0$. In this case, the

Table 1: Iterative Phase-Only Signal Reconstruction

	NUMBER OF ITERATIONS	x[0]	x[1]	x[2]	x[3]	x[4]	x[5]	x[6]	x[7]	SQUARED ERROR
Standard Iterative Procedure	10	4.000	1.535	-8.451	2.733	4.428	6.061	11.907	-4.329	$2.552 \cdot 10$
	100	4.000	1.972	-9.886	3.152	4.743	6.615	13.371	-5.228	$1.107 \cdot 10$
	1000	4.000	2.015	-10.904	4.757	4.110	5.239	14.825	-5.935	$1.729 \cdot 10^{-1}$
1st-Order Adaptive Acceleration	10	4.000	1.895	-9.545	2.942	4.749	6.648	12.995	-5.003	$1.465 \cdot 10$
	30	4.000	2.093	-10.533	3.730	4.599	6.289	14.129	-5.675	4.725
	50	4.000	1.998	-10.901	4.858	4.027	5.077	14.840	-5.949	$6.492 \cdot 10^{-2}$
	ORIGINAL SEQUENCE	4.0	2.0	-11.0	5.0	4.0	5.0	14.0	-6.0	

solution to the phase-only problem can be expressed as

$$\underline{x} = \beta \begin{bmatrix} 1 \\ \text{------} \\ S^{-1}\,\underline{b} \end{bmatrix} \qquad (3)$$

where $\underline{x}$ represents the vector of elements of $x[n]$ and $\beta = x[0]$ is the arbitrary scaling factor.

Compared with the iterative algorithm, the closed form solution guarantees the correct solution without any iterations and provides the added flexibility of non-uniform sampling of the phase function. The primary disadvantage, however, is that the inverse of an $(N-1)x(N-1)$ matrix must be computed which, as N gets large, becomes more difficult and subject to serious round-off errors.

Finally, an example of the closed form solution is shown in Figure 2 in which a 16x16 2-D array was reconstructed from its phase using the projection-slice approach. The reconstructed image, expanded for visual purposes by a zero-order hold, is indistinguishable from the original.

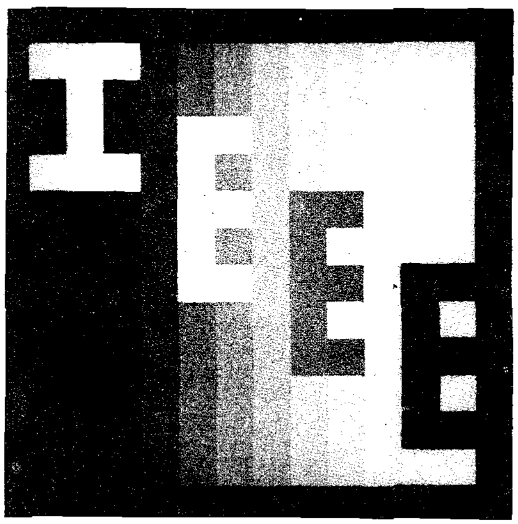

Figure 2: Phase-Only Signal Reconstruction Using Closed Form Solution

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