RANDOMIZED SINC INTERPOLATION OF NONUNIFORM SAMPLES

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ABSTRACT

It is well known that a bandlimited signal can be uniquely determined from nonuniformly spaced samples, provided that the average sampling rate exceeds the Nyquist rate. However, reconstruction of the continuous-time signal from nonuniform samples is more difficult than from uniform samples. This paper develops and compares simpler approximate methods for signal reconstruction from nonuniform samples.

1. INTRODUCTION

The most common form of sampling used in the context of discrete-time processing of continuous-time signals is uniform sampling. For a bandwidth-limited signal $x(t)$ whose Fourier spectrum contains no component at or above the frequency Ω_c the well-known Nyquist-Shannon sampling theorem states that the signal is uniquely determined by its values at an infinite set of sample points spaced at $T_N = \pi/\Omega_c$ apart. Specifically, $x(t)$ is represented in terms of its uniform samples as

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_N)\cdot h(t-kT_N) \quad (1)$$

where $h(t) = sinc(\pi/T_N \cdot t)$ [1].

Various extensions of the uniform sampling theorem are well known (see, for example Papoulis [1]). In [2] some special nonuniform sampling processes are examined in detail and generalized sampling theorems are obtained. Yao and Thomas [3] discuss extensions of the uniform sampling expansion and establish that a bandlimited signal can be uniquely determined from nonuniform samples, provided that the average sampling rate exceeds the Nyquist rate. However, in contrast to uniform sampling, reconstruction of the continuous-time signal from nonuniform samples using direct interpolation is computationally difficult. Several alternative methods for reconstruction from nonuniform samples have been previously suggested, such as iterative algorithms (e.g. [4]) which are also computationally demanding and have potential issues of convergence. In [5] the bandlimited assumption is replaced by a smoothness assumption and the use of polynomial filtering for reconstruction of nonuniformly sampled signal is considered. Marvasti [6] suggests a method to recover a bandlimited signal from an nth-order-hold version of irregular samples. In [7] the transposed Farrow structure is used for converting the nonuniformly sampled sequence into uniform one. Papoulis [8] suggests a nonlinear transformation of the nonuniform grid into a uniform grid and develops an approximate reconstruction of a signal from its nonuniform samples. A method of designing FIR filters whose input signals are sampled irregularly due to clock jitter is presented in [9]. Methods for reconstruction when the nonuniform sampling pattern has a periodically recurring structure have also been proposed [10],[11].

In this paper we treat nonuniform samples as a stochastic perturbation from a uniform grid. With this approach, developed in section 2, the characteristic function of the perturbation error plays a role similar to that of an anti-aliasing filter. In section 3, using the model in section 2, several approaches are suggested and analyzed for approximate reconstruction from nonuniform samples. These methods, based on the uniform sampling reconstruction of eq. (1) and its Taylor's series expansion, lead to sinc interpolation of the nonuniform samples treated as though they are on a uniform grid, and alternatively sinc interpolation applied to the samples on the nonuniform grid. In section 4 these methods are compared in terms of their mean squared error (MSE). A generalized reconstruction method is also proposed in section 4, which incorporates both methods of section 3 as special cases. This generalized method consists of locating the samples randomly around the uniform grid with the characteristics of the random perturbations designed to minimize the reconstruction error. Section 5 suggests applying a Wiener filter to improve the mean squared error obtained by this randomized sinc interpolation method.

[1] where we use throughout the historical unnormalized definition of the sinc function, i.e., $sinc(x) \triangleq \frac{sin(x)}{x}$

2. STOCHASTIC PERTURBATION MODEL OF NONUNIFORM SAMPLING

We consider $x(t)$ to be a continous-time zero-mean wide sense stationary random process with autocorrelation function $R_x(\tau)$ and power spectral density (PSD) $S_x(\Omega)$ which is zero for $|\Omega| \geq \Omega_c$. We denote by $\tilde{x}[n]$ a nonuniform sequence of samples of $x(t)$, i.e.,

$$\tilde{x}[n] = x(t_n) \quad (2)$$

where $\{t_n\}$ represent a nonuniform grid which we model as random perturbations of a uniform grid , i.e.,

$$t_n = nT + \xi_n \ . \quad (3)$$

T denotes the nominal sampling interval. ξ_n is characterized as an i.i.d. sequence of random variables independent of $x(t)$ with probability density function (pdf) $f_\xi(\xi)$ and characteristic function $\Phi_\xi(\Omega) = \int_{-\infty}^{\infty} f_\xi(\xi')e^{j\Omega\xi'}d\xi'$. The objective is to reconstruct $x(t)$ from its nonuniform samples $\tilde{x}[n]$.

We first show that with respect to second-order statistics, $\tilde{x}[n]$ can equivalently be represented by the sequence $z[n]$ in figure (1) where the system $\Phi_\xi(\Omega)$ has frequency response equal to the Fourier transform of $f_\xi(\xi)$ and $v[n]$ is zero-mean additive white noise, uncorrelated with $x(t)$, with PSD $S_v = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)(1-|\Phi_\xi(\Omega)|^2)d\Omega$.

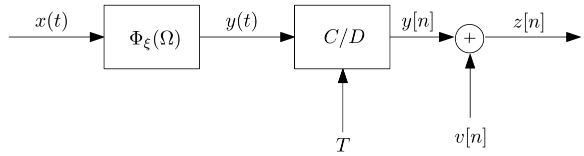

Figure 1: A second-order statistics model for nonuniform sampling

For the system of figure (1) it is straight forward to show that

$$R_z[n,n-k] = \begin{cases} \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)d\Omega & k=0 \\ \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)|\Phi_\xi(\Omega)|^2 e^{j\Omega Tk}d\Omega & k\neq 0 \end{cases} \quad (4)$$

and that

$$E(z[n]x(t)) = R_x(\tau) * f_\xi(\tau)|_{\tau=nT-t}. \quad (5)$$

To show that eq. (4) is identical to $R_{\tilde{x}}[n,n-k]$ and eq. (5) is identical to $E(\tilde{x}[n]x(t))$, we evaluate the autocorrelation of $\tilde{x}[n]$ and the cross-correlation between $\tilde{x}[n]$ and $x(t)$. Specifically, with both $x(t_n)$ and ξ_n as random variables, the autocorrelation function of $\tilde{x}[n]$ is given by,

$$\begin{aligned} & R_{\tilde{x}}[n,n-k] = E\{\tilde{x}[n]\tilde{x}[n-k]\} = \\ & E\{x(nT+\xi_n)x((n-k)T+\xi_{n-k})\} = \\ & E\{R_x(kT+\xi_n-\xi_{n-k})\} \end{aligned} \quad (6)$$

Expressing (6) in terms of $S_x(\Omega)$ we obtain,

$$R_{\tilde{x}}[n,n-k] = E\left(\frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)e^{j\Omega(kT+\xi_n-\xi_{n-k})}d\Omega\right) =$$

$$\begin{cases} \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)d\Omega & k=0 \\ \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)|\Phi_\xi(\Omega)|^2 e^{j\Omega Tk}d\Omega & k \neq 0 \end{cases} \quad (7)$$

which is identical to eq. (4).
The input-output cross-correlation is given by

$$\begin{aligned} E(\tilde{x}[n]x(t)) &= E(x(nT+\xi_n)x(t)) = E(R_x((nT-t)+\xi_n)) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\Omega)\Phi_\xi(\Omega)e^{j\Omega(nT-t)}d\Omega \\ &= R_x(\tau) * f_\xi(\tau)|_{\tau=nT-t} \end{aligned} \quad (8)$$

which is equivalent to the input-output cross-correlation in the system in figure (1), i.e. to eq. (5).
Eq. (7) can also equivalently be written in terms of the continuous autocorrelation function of $x(t)$ as

$$\begin{aligned} R_{\tilde{x}}[k] &= R_{\tilde{x}}[n,n-k] = R_x(t) * f_\xi(t) * f_\xi(-t)|_{t=kT} \\ &+ \left(R_x(0) - R_x(t) * f_\xi(t) * f_\xi(-t)|_{t=0}\right) \cdot \delta[k] \ . \end{aligned} \quad (9)$$

Transforming to the frequency domain, we obtain

$$\begin{aligned} S_{\tilde{x}}(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} S_x\left(\frac{\omega-2\pi k}{T}\right)\left|\Phi_\xi\left(\frac{\omega-2\pi k}{T}\right)\right|^2 \\ &+ \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)\left(1-|\Phi_\xi(\Omega)|^2\right)d\Omega \end{aligned} \quad (10)$$

Uniform sampling is a special case for which $f_\xi(\xi) = \delta(\xi)$ and $\Phi_\xi(\Omega) = 1$. In this case, as expected,

$$S_{\tilde{x}}(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} S_x\left(\frac{\omega-2\pi k}{T}\right) \quad (11)$$

which is the power spectral density of the sequence of uniform samples $x[n] = x(nT)$. The effect of timing error on the power spectrum indicated in eq. (10) was first shown by Akaike in [12].

The structure of figure (1) suggests that with respect to second-order statistics, nonuniform sampling with stochastic perturbations can be modeled as uniform sampling of the signal pre-filtered by the Fourier transform of the pdf of the sampling perturbation. Correspondingly, the pdf $f_\xi(\xi)$ can be designed subject to the constraints on $f_\xi(\xi)$ as a probability density function so that $\Phi_\xi(\Omega)$ acts as an equivalent anti-aliasing LPF in figure (1). Of course the stochastic perturbation still manifests itself through the additive white noise $v[n]$ in figure (1). Thus, figure (1) suggests that aliasing can be traded off with uncorrelated white noise by appropriate design of the pdf of the sampling perturbation.

An interesting consequence of eq. (10) is that it offers the potential in the context of nonuniform sampling to resolve whether or not the signal has been undersampled. To illustrate, consider the signal

$$x(t) = A \cdot cos(\Omega_0 t + \theta) \quad (12)$$

where $\theta \sim u[-\pi,\pi]$ and A and Ω_0 are deterministic unknown parameters. The PSD of $x(t)$ is given by

$$S_x(\Omega) = \frac{\pi A^2}{2} \cdot (\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)) \quad (13)$$

By substituting $S_x(\Omega)$ from (13) into (10) we obtain

$$\begin{aligned} S_{\tilde{x}}(e^{j\omega}) &= \frac{1}{2}A^2\left(1-|\Phi_\xi(\Omega_0)|^2\right) + \frac{\pi}{2}A^2 \cdot |\Phi_\xi(\Omega_0)|^2 \cdot \\ &\cdot \sum_{k=-\infty}^{\infty}(\delta(\omega-\Omega_0 T-2\pi k) + \delta(\omega+\Omega_0 T-2\pi k)) \end{aligned} \quad (14)$$

where the PSD of the uniformly sampled signal $x[n] = x(nT)$ is given by,

$$S_x(e^{j\omega}) = \frac{\pi A^2}{2} \cdot \sum_{k=-\infty}^{\infty}(\delta(\omega-\Omega_0 T-2\pi k) + \delta(\omega+\Omega_0 T-2\pi k))$$

Given $S_x(e^{j\omega})$, we can solve for A^2 and for Ω_0. However, the solution for Ω_0 is not unique if the sampling rate is not guaranteed to be above the Nyquist rate. In the case of nonuniform sampling, however, the noise floor as well as the attenuation factor depends on the continuous frequency Ω_0 through $\Phi_\xi(\cdot)$ which allows us to uniquely solve for A^2 and $|\Phi_\xi(\Omega_0)|$. Consequently, if sampling is done nonuniformly, we can determine whether the signal was undersampled and also the value of Ω_0 for well behaved characteristic functions.

To generalize this idea, assume that $x(t)$ is sampled nonuniformly with an average rate that exceeds the Nyquist rate. Then,

$$S_{\tilde{x}}(e^{j\pi}) = S_v = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)\left(1-|\Phi_\xi(\Omega)|^2\right)d\Omega \quad (15)$$

and

$$S_{\tilde{x}}(e^{j\omega}) - S_{\tilde{x}}(e^{j\pi}) = \frac{1}{T}\cdot S_x\left(\frac{\omega}{T}\right)\left|\Phi_\xi\left(\frac{\omega}{T}\right)\right|^2 \quad \forall \quad |\omega| < \pi \quad (16)$$

Therefore, if $\Phi_\xi(\Omega) \neq 0$ for all $\Omega < \Omega_c$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(S_{\tilde{x}}(e^{j\omega}) - S_{\tilde{x}}(e^{j\pi})\right)\frac{\left(1-|\Phi_\xi(\frac{\omega}{T})|^2\right)}{|\Phi_\xi(\frac{\omega}{T})|^2}d\omega = S_v \quad (17)$$

Thus, comparing the left-hand side of (17) with $S_{\tilde{x}}(e^{j\pi})$, we can tell whether $x(t)$ was undersampled or not.

3. APPROXIMATE RECONSTRUCTION FROM NONUNIFORM SAMPLES

In this section we suggest several simplified but approximate approaches to reconstruction based on the model and analysis discussed in section 2. It will be assumed throughout the rest of the paper that the average sampling rate exceeds the Nyquist rate and consequently that the direct interpolation formula would result in exact reconstruction. $x(t)$ can in general be expressed in terms of the uniform samples $x_k = x(kT_N)$ through eq. (1). Consequently, $x(t_n)$ can be expressed as

$$x(t_n) = \sum_{k=-\infty}^{\infty} x_k \cdot h(t_n - kT_N) \quad (18)$$

With $t_n = nT + \xi_n$ where T is the nominal sampling interval and expanding eq. (18) in a Taylor's series in t_n around $t_n = nT$ we obtain the M^{th}-order approximation

$$\widetilde{x}_M(t_n) = \sum_{k=-\infty}^{\infty} x_k \cdot \left(\sum_{p=0}^{M} \frac{\xi_n{}^p}{p!} \cdot h^{(p)}(nT - kT_N) \right) \tag{19}$$

where $h^{(p)}(t) = \frac{d^p h(t)}{dt^p}$.
Our basic approach is to treat the parameters x_k as deterministic and determine their values to minimize the conditional mean squared error between $x(t_n)$ and $\widetilde{x}_M(t_n)$ as specified in eq. (19). Specifically, we choose x_k to minimize

$$E\left\{ \sum_{n=-\infty}^{\infty} \left(x(t_n) - \sum_{k=-\infty}^{\infty} x_k \cdot \sum_{p=0}^{M} \frac{\xi_n{}^p}{p!} \cdot h^{(p)}(nT - kT_N) \right)^2 \middle| \{x(t_n)\} \right\}$$

Differentiating w.r.t x_l and setting to zero we obtain

$$\sum_{k=-\infty}^{\infty} x_k \cdot \sum_{p,q=0}^{M} \sum_{n=-\infty}^{\infty} \frac{E\left(\xi_n{}^{p+q} \middle| \{x(t_n)\}\right)}{p! \cdot q!} \cdot h^{(p)}(nT - kT_N) \cdot h^{(q)}(nT - lT_N)$$
$$= \sum_{n=-\infty}^{\infty} x(t_n) \cdot \left(\sum_{p=0}^{M} \frac{E\left(\xi_n{}^p \middle| \{x(t_n)\}\right)}{p!} \cdot h^{(p)}(nT - lT_N) \right) \quad \forall l \tag{20}$$

which under the independence assumption of $\{\xi_n\}$ and $x(t)$ becomes

$$\sum_{k=-\infty}^{\infty} x_k \cdot \sum_{p=0}^{M} \sum_{q=0}^{M} \frac{m_{p+q}}{p! \cdot q!} \cdot \left(\sum_{n=-\infty}^{\infty} h^{(p)}(nT - kT_N) \cdot h^{(q)}(nT - lT_N) \right)$$
$$= \sum_{n=-\infty}^{\infty} x(t_n) \cdot \left(\sum_{p=0}^{M} \frac{m_p}{p!} \cdot h^{(p)}(nT - lT_N) \right) \tag{21}$$

where $m_p = E(\xi_n{}^p)$ is the p^{th}-order moment of ξ_n.
Using the equality

$$\sum_{n=-\infty}^{\infty} h^{(p)}(nT - kT_N) \cdot h^{(q)}(nT - lT_N) = \frac{T_N}{T} \cdot (-1)^q \cdot h^{(p+q)}((l-k)T_N)$$

the left side of (21) is the convolution of x_k with the sequence

$$c[k] = \frac{T_N}{T} \cdot \sum_{p=0}^{M} \sum_{q=0}^{M} \frac{m_{p+q}}{p! \cdot q!} \cdot (-1)^q \cdot h^{(p+q)}(kT_N) \tag{22}$$

Treating the sequence x_k as uniform samples in reconstructing $x(t)$ will then obtain

$$\hat{x}_M(t) = \hat{x}_0(t) * w_M(t) \tag{23}$$

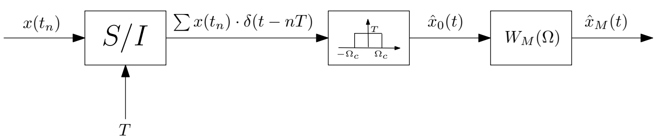

Figure 2: M^{th}-order approximate reconstruction

where $\hat{x}_0(t)$ is the optimal zeroth-order approximation,

$$\hat{x}_0(t) = \sum_{n=-\infty}^{\infty} (T/T_N) \cdot x(t_n) \cdot h(t - nT) \tag{24}$$

and

$$W_M(\Omega) = \frac{\sum_{p=0}^{M} \frac{m_p}{p!} (-j\Omega)^p}{\sum_{p=0}^{M} \sum_{q=0}^{M} \frac{m_{p+q}}{p! \cdot q!} \cdot (-1)^q \cdot (j\Omega)^{p+q}} \quad |\Omega| < \Omega_c \tag{25}$$

This then corresponds to treating the nonuniform samples as being on a uniform grid and reconstructing $x(t)$ using the filter $T \cdot W_M(\Omega)$. Note that as M increases, an increasingly higher-order statistics of ξ_n are needed for this optimal reconstruction.

An alternative approach to the use of the Taylor's expansion is to choose the coefficients in eq. (18) to minimize the conditional mean squared error between the actual nonuniform samples and the nonuniform samples that would result from the values x_k treated on a uniform grid, i.e. choose x_k to minimize

$$E\left\{ \sum_{n=-\infty}^{\infty} \left(x(t_n) - \sum_{k=-\infty}^{\infty} x_k \cdot h(t_n - kT_N) \right)^2 \middle| \{x(t_n)\} \right\} \tag{26}$$

This minimization results in the equations:

$$\sum_{n=-\infty}^{\infty} x(t_n) \cdot E\{h(t_n - lT_N)\} = \tag{27}$$
$$\sum_{k=-\infty}^{\infty} x_k \cdot E\left\{ \sum_{n=-\infty}^{\infty} h(t_n - kT_N) h(t_n - lT_N) \right\} \quad \forall l$$

By using the relation

$$E\left\{ \sum_{n=-\infty}^{\infty} h(t_n - kT_N) h(t_n - lT_N) \right\} = \frac{T_N}{T} \cdot \delta[l-k] \tag{28}$$

eq. (27) becomes

$$\sum_{n=-\infty}^{\infty} x(t_n) \cdot E\{h(t_n - lT_N)\} = \frac{T_N}{T} \cdot x_l \quad \forall l \tag{29}$$

Taking the expectation on the left hand side of eq. (29) will lead to the reconstruction suggested in (23) with $M \to \infty$, i.e.,

$$W_M(\Omega) = \Phi_{\xi}^{*}(\Omega) \quad |\Omega| < \Omega_c \tag{30}$$

As an alternative, we suggest replacing $E\{h(t_n - lT_N)\}$ with $h(t_n - lT_N)$ in which case

$$\hat{x}_k \quad = \quad \sum_{n=-\infty}^{\infty} (T/T_N) \cdot x(t_n) \cdot h(t_n - kT_N) \tag{31}$$

and

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} (T/T_N) \cdot x(t_n) \cdot h(t - t_n) \tag{32}$$

This then corresponds to reconstruction using sinc interpolation applied to the samples on the nonuniform grid as represented in Fig. (3). This approximation will be referred to as nonuniform sinc interpolation.

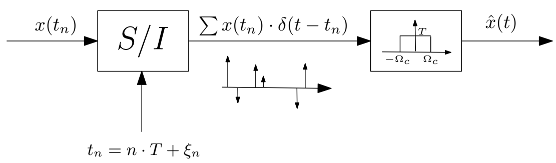

Figure 3: Nonuniform Sinc Interpolation

4. PERFORMANCE OF UNIFORM, NONUNIFORM AND RANDOMIZED SINC INTERPOLATION

In this section, we analyze the performance of the methods suggested in the previous section with respect to their mean squared error. We denote by $e_M^U(t)$ the error between $x(t)$ and $\hat{x}_M(t)$ defined in (23). Then, it can be shown that the MSE is

$$\begin{aligned}\sigma^2_{e_M^U} &= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)\cdot\Big\{|1-\Phi_\xi(\Omega)\cdot W_M(\Omega)|^2+ \\ &+ \rho\cdot\overline{W_M}\cdot\left(1-|\Phi_\xi(\Omega)|^2\right)\Big\}d\Omega \qquad (33)\end{aligned}$$

where $\rho = T/T_N \leq 1$ and $\overline{W_M} = \frac{1}{2\Omega_c}\cdot\int_{-\Omega_c}^{\Omega_c}|W_M(\Omega)|^2 d\Omega$. For the uniform sinc interpolation, which corresponds to $M=0$, the MSE is

$$\sigma^2_{e^U} = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)\cdot\underbrace{\left\{|1-\Phi_\xi(\Omega)|^2+\rho\cdot\left(1-|\Phi_\xi(\Omega)|^2\right)\right\}}_{\triangleq G_U(\Omega)}d\Omega \quad (34)$$

Similarly, denote by $e^{\mathcal{U}}(t)$ the reconstruction error of the nonuniform sinc interpolation method defined in (32). Then,

$$\sigma^2_{e^{\mathcal{U}}} = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)\cdot\underbrace{\rho\cdot\left(1-\frac{1}{2\Omega_c}\int_{\Omega-\Omega_c}^{\Omega+\Omega_c}|\Phi_\xi(\Omega')|^2 d\Omega'\right)}_{\triangleq G_{\mathcal{U}}(\Omega)}d\Omega \quad (35)$$

It can be shown and is intuitively reasonable that both the uniform and nonuniform sinc interpolation methods attain zero MSE if and only if the samples had been generated uniformly, i.e., $\tilde{x}[n]=x(nT)$. Also, as is clear from eqns (34),(35) the performance of both methods depends on the spectrum of the continuous-time signal $x(t)$ as well as on the characteristic function $\Phi_\xi(\Omega)$ of the perturbations error, which can be designed to reduce the MSE. The MSE depends also on the oversampling ratio $r=1/\rho=T_N/T$. However, while the performance of both methods improves as r increases, only nonuniform sinc interpolation approaches zero MSE when r approaches infinity. This occurs since nonuniform sinc interpolation maintains the correct sample positions in time whereas uniform sinc interpolation does not.

For the purpose of comparison between the two methods, we consider two cases. The first is the case of small perturbations with zero mean, for which in the region $|\Omega|<2\Omega_c$, $\Phi_\xi(\Omega)$ can be well approximated by the second-order Taylor's expansion

$$\Phi_\xi(\Omega)\approx 1-\frac{1}{2}\sigma_\xi^2\Omega^2 \quad (36)$$

with the variance σ_ξ^2 of ξ_n assumed to be small enough relative to T so that (36) holds. Substituting (36) into (34) and (35) yields

$$\sigma^2_{e^{\mathcal{U}}}/\sigma^2_{e^U} \approx \left(1+{\Omega_c}^2/3B_x\right) \quad (37)$$

where B_x denotes the bandwidth of $x(t)$ defined as

$$B_x = \int_{-\Omega_c}^{\Omega_c}\Omega^2\cdot\left(\frac{S_x(\Omega)}{\int_{-\Omega_c}^{\Omega_c}S_x(\Omega')d\Omega'}\right)d\Omega \quad (38)$$

We see from (37) that independent of the detailed characteristics of the perturbation or the signal spectrum, as long as the perturbations around the uniform grid are small enough so that (37) holds, it is better to reconstruct the signal using uniform sinc interpolation, even though uniform sinc interpolation uses only the nominal rather than actual sampling times, and therefore uses less information than the nonuniform sinc interpolation method.

The second case to be examined is that of uniformly distributed peturbations, i.e.,

$$\xi_n \sim u[-\Delta_\xi,\Delta_\xi] \leftrightarrow \Phi_\xi(\Omega)=sinc(\Delta_\xi\Omega) \quad (39)$$

For the case in which the nominal sampling rate equals the Nyquist rate, i.e., $T=T_N$, $G_U(\Omega)$ and $G_{\mathcal{U}}(\Omega)$ defined in (34) and (35) becomes

$$\begin{aligned}G_U(\Omega) &= 2(1-sinc(\Delta_\xi\Omega)) \;\; |\Omega|<\Omega_c \qquad (40)\\ G_{\mathcal{U}}(\Omega) &= 1-\frac{1}{2\Omega_c}\int_{\Omega-\Omega_c}^{\Omega+\Omega_c}sinc^2(\Delta_\xi\Omega')d\Omega' \quad |\Omega|<\Omega_c\end{aligned}$$

From (40) it follows that for $\Delta_\xi<0.34T$,

$$G_U(\Omega)<G_{\mathcal{U}}(\Omega)\quad\forall\;|\Omega|<\Omega_c \quad (41)$$

Consequently for $\Delta_\xi<0.34T$, uniform sinc interpolation always achieves a lower MSE than nonuniform sinc interpolation independent of the input spectrum. However, when $\Delta_\xi>0.34T$, the relative performance of these two methods strongly depends on the spectrum of the input signal.

None of the methods suggested above is universally preferable for all signals. We next suggest a generalized method which will be referred to as randomized sinc interpolation. This method suggests applying sinc interpolation to the samples on the nonuniform grid $\tilde{t}_n=nT+\zeta_n$, where ζ_n is another iid sequence of random variables independent of $x(t)$ and for which ζ_n is independent of ξ_k for $n\neq k$. The reconstruction then takes the form

$$\hat{x}(t)=\sum_{n=-\infty}^{\infty}(T/T_N)\cdot x(t_n)\cdot h(t-\tilde{t}_n) \quad (42)$$

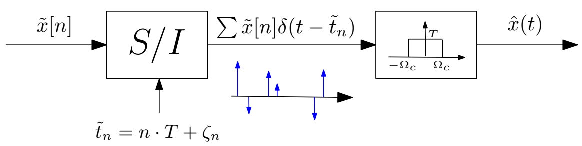

Figure 4: Randomized Sinc Interpolation

The uniform sinc interpolation as well as the nonuniform sinc interpolation discussed above can be treated as special cases of this generalized method with $\zeta_n=0$ and $\zeta_n=\xi_n$, respectively. Allowing ζ_n to have a partial correlation with ξ_n is a further generalization.

Provided that the average sampling rate exceeds the Nyquist rate, it can be shown that with respect to second-order statistics, nonuniform sampling discussed in section 2 followed by the randomized reconstruction method suggested in figure (4) is equivalent to the following system

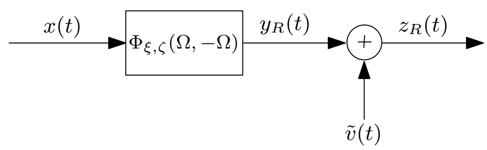

Figure 5: A second-order statistics model for nonuniform sampling followed by randomized sinc interpolation reconstruction

where $\Phi_{\xi,\zeta}(\Omega_1,\Omega_2)$ is the joint characteristic function of ξ_n and ζ_n, defined as the Fourier transform of their joint pdf $f_{\xi,\zeta}(\xi,\zeta)$ and $\tilde{v}(t)$ is zero-mean additive colored noise, uncorrelated with $x(t)$, with PSD

$$S_{\tilde{v}}(\Omega)=\frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c}S_x(\Omega')(1-|\Phi_{\xi,\zeta}(\Omega',-\Omega)|^2)d\Omega' \quad |\Omega|<\Omega_c. \quad (43)$$

The corresponding MSE of the randomized sinc interpolation method is given by

$$\sigma_{e^R}^2 = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} S_x(\Omega)\cdot\left\{\left|1-\Phi_{\xi,\zeta}(\Omega,-\Omega)\right|^2 + \right.$$
$$\left. +\rho\cdot\left[1-\frac{1}{2\Omega_c}\int_{-\Omega_c}^{\Omega_c}|\Phi_{\xi,\zeta}(\Omega,-\Omega_1)|^2 d\Omega_1\right]\right\}d\Omega \qquad (44)$$

Designing ζ_n to optimize the MSE is done through the joint characteristic function $\Phi_{\xi,\zeta}(\Omega_1,\Omega_2)$ keeping the constraint $\Phi_{\xi,\zeta}(\Omega,0)=\Phi_\xi(\Omega)$. To illustrate, we will assume that $(\xi_n,\zeta_n)\sim N(0,0,\sigma_\xi^2,\sigma_\zeta^2,\rho_{\xi\zeta})$ and find σ_ζ^2 and $\rho_{\xi\zeta}$ to optimize the MSE. For the optimal solution, $\rho_{\xi\zeta}=1$ and σ_ζ grows with the bandwidth of the input signal as shown in figure (6).

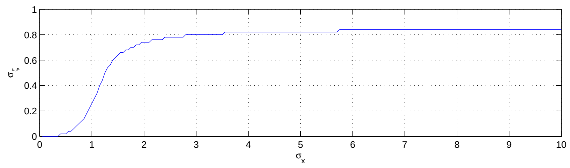

Figure 6: The optimal std σ_ζ of ζ_n as a function of the spread σ_x of the spectrum of $x(t)$ for $\sigma_\xi = 1$

This can be achieved by choosing $\zeta_n = (\sigma_\zeta/\sigma_\xi)\cdot\xi_n$. Therefore, for low bandwidth signals, $\sigma_\zeta = 0$ and the optimal reconstruction method is the uniform sinc interpolation. As the bandwidth of the input signal is increased, σ_ζ is increased and as a result the samples are positioned closer to their original location but still with tendency towards the uniform grid due to the optimality of this reconstruction method.

5. WIENER FILTERING

The methods discussed in section 4 can be further improved by pre-filtering and post-filtering as indicated in figure (7). The pre-filter prior to sampling is used to shape the spectrum and thus reduce the noise. Also, the discrete-time signal after sampling, $\tilde{x}[n]$, can be pre-processed prior to reconstruction. Finally, post-processing of the signal $\hat{x}(t)$ after reconstruction can be obtained to reduce the MSE.

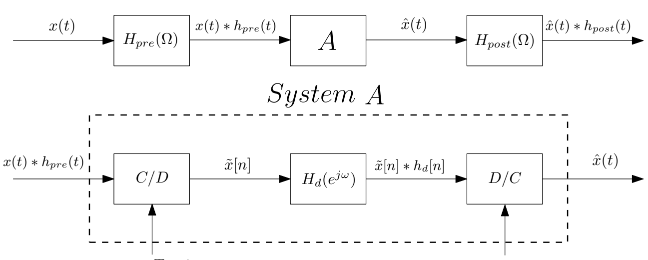

Figure 7: Improved Randomized Sinc Interpolation

We consider here only the design of the post-filtering $H_{post}(\Omega)$. Minimizing the MSE with respect to $H_{post}(\Omega)$ will then correspond to the use of a non-causal Wiener filter

$$H_{post}(\Omega) = \frac{S_{x\hat{x}}(\Omega)}{S_{\hat{x}}(\Omega)} \qquad (45)$$

where $S_{x\hat{x}}(\Omega) = S_x(\Omega)\cdot\Phi^*_{\xi,\zeta}(\Omega,-\Omega)$ and

$$S_{\hat{x}}(\Omega) = S_x(\Omega)\cdot\left|\Phi_{\xi,\zeta}(\Omega,-\Omega)\right|^2 + S_{\tilde{v}}(\Omega) \quad |\Omega|<\Omega_c \qquad (46)$$

with $S_{\tilde{v}}(\Omega)$ defined in (43). Consequently,

$$H_{post}(\Omega) = \frac{S_x(\Omega)\cdot\Phi^*_{\xi,\zeta}(\Omega,-\Omega)}{S_x(\Omega)\cdot|\Phi_{\xi,\zeta}(\Omega,-\Omega)|^2+S_{\tilde{v}}(\Omega)} \qquad |\Omega|<\Omega_c \quad (47)$$

The corresponding MSE is given by

$$\sigma_{e^W}^2 = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c}\frac{S_x(\Omega)\cdot S_{\tilde{v}}(\Omega)}{S_x(\Omega)\cdot|\Phi_{\xi,\zeta}(\Omega,-\Omega)|^2+S_{\tilde{v}}(\Omega)}d\Omega \qquad (48)$$

As indicated previously, the uniform sinc interpolation correponds to the special case of $\zeta_n = 0$. In this case, applying the filter $H_{post}(\Omega)$ on $\hat{x}(t)$ is equivalent to applying the discrete-time filter

$$H_d(e^{j\omega}) = H_{post}\left(\frac{\omega}{T}\right) = \frac{S_x(\frac{\omega}{T})\cdot\Phi^*_\xi(\frac{\omega}{T})}{S_x(\frac{\omega}{T})\cdot|\Phi_\xi(\frac{\omega}{T})|^2+S_{\tilde{v}}(0)} \quad |\omega|<\pi \quad (49)$$

to the nonuniform samples $\tilde{x}[n]$ prior to the reconstruction. This filter can be interpreted as the linear minimum mean squared error estimator of the uniform samples $x[n]$ from the nonuniform samples $\tilde{x}[n]$. That is, the best filter to apply to the nonuniform samples prior to the uniform sinc interpolation is one that estimates the uniform samples.

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