MULTISCALE ANALYSIS OF FRACTAL POINT PROCESSES AND QUEUES

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ABSTRACT

Using a powerful, recently-developed multiscale representation for fractal point processes, we characterize these processes under fundamental transformations which arise in a variety of important applications such as data network traffic modeling. Insightful distributional results are obtained, including the interarrival density for a fractal point process subject to random erasure, the counting process distribution for the superposition of fractal point processes, and the steady-state customer distribution in queues with self-similar customer arrivals. Interpretations and implications of these results for applications are also discussed.

1. INTRODUCTION

Fractal point processes—self-similar distributions of events in time or space—are important models in a wide range of signal processing applications. They are well-matched to, for example, stellar and planetary distributions, auditory neuron firing patterns in mammals, distributions of human and biological populations, and vehicular traffic within cities [1][2][3]. They also constitute particularly promising models for data traffic on a wide range of packet-switched networks, from local ethernet links to the internet as a whole [4][5]. Of particular interest is the family of fractal renewal processes formally defined in [6], which are generalized renewal processes in the following sense: when viewed over a finite window with finite resolution, the observations constitute a renewal process, with interarrivals X governed by the power-law (Pareto) density

$$f_X(x) = \frac{\sigma_x^2}{x^\gamma}, \qquad 0 < \underline{x} < x < \overline{x} < \infty. \tag{1}$$

In (1), γ is called the shape parameter and is directly related to the fractal dimension of the point process, $\underline{x}$ and $\overline{x}$ are determined by the resolution and size of the observation window respectively, and σ_x^2 is a normalization factor.

While traditional signal processing methods have yielded limited results for fractal point processes, multiscale paradigms have proven to be natural for these signal models. In particular, based on a finite-scale framework, we have recently developed a number of practical estimation and classification algorithms for fractal renewal processes [6]. In the present paper, we extend our multiscale techniques to characterize fractal renewal processes under important classes of transformations such as random erasure, superposition, and queueing. Moreover, we demonstrate that key issues such as random incidence can be readily incorporated in these studies. Among their many applications, our results have potentially important implications in the design and management of networks.

This work has been supported in part by ARPA, monitored by ONR under Contract No. N00014-93-1-0686, NSF under Grant No. MIP-9502885, and ONR under Grant No. N0014-95-1-0834.

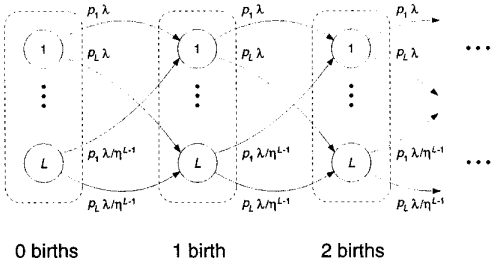

Figure 1: *A multiscale pure-birth process; dashed boxes denote conceptual partitioning into superstates.*

2. MULTISCALE PURE-BIRTH PROCESS

The multiscale pure-birth process (see Fig. 1) provides a novel perspective for our multiscale framework of [6], and furnishes the foundation for our present exposition. Generalizing the well-known pure-birth process (see, e.g., [7]), the state space of this Markov process consists of a set of "superstates," each of which corresponds to a certain number of births. Included in a superstate is a set of states corresponding to the scales in our finite-scale framework. Hence, each state is naturally indexed with an ordered pair of integers (i, j), where $i \geq 0$ is the superstate index, and j is the scale index within each superstate, which ranges from 1 to the number of scales L. The state distribution at any time t is denoted by $P_{i,j}(t)$.

In accordance with our multiscale framework, the mean departure rate from every state (i, j) is determined by its scale and is of the form λ/η^{j-1}, where λ is the rate of the finest-scale constituent and $\eta > 1$ is called the scale in-

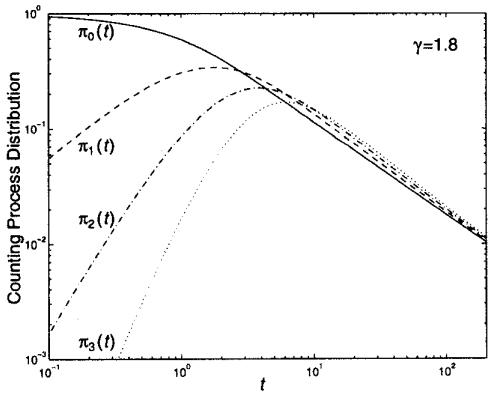

Figure 2: *Arrival-observed counting process distribution for a fractal renewal process.*

crement. Upon departure, every state j' of the succeeding superstate $i+1$ can be directly reached, and is chosen with probability $p_{j'} = \sigma^2 q^{j'-1}$, where $q \triangleq \eta^{1-\gamma}$, and σ^2 is a normalization term. From the state-space description, it is straightforward to set up the system of forward Kolmogorov equations [7], which governs the dynamics of the multiscale pure-birth process. Specifically, we have

$$\frac{d}{dt}\mathbf{P}_0(t) = -\mathbf{\Lambda}\mathbf{P}_0(t) \quad (2a)$$

$$\frac{d}{dt}\mathbf{P}_i(t) = -\mathbf{\Lambda}\mathbf{P}_i(t) + \mathbf{Q}\mathbf{B}^{\mathrm{T}}\mathbf{P}_{i-1}(t), \quad i > 0 \quad (2b)$$

where $\mathbf{P}_i(t) \triangleq (P_{i,1}(t), \ldots, P_{i,L}(t))^{\mathrm{T}}$ is the probability distribution in superstate i, $\mathbf{Q} \triangleq (\sigma^2, \sigma^2 q, \ldots, \sigma^2 q^{L-1})^{\mathrm{T}}$ is a vector of the choice probabilities, $\mathbf{B} \triangleq (\lambda, \lambda/\eta, \ldots, \lambda/\eta^{L-1})^{\mathrm{T}}$ is a vector of the constituent rates, and $\mathbf{\Lambda} \triangleq \mathrm{diag}(\mathbf{B})$ is a diagonal matrix with the constituent rates along its main diagonal. Using standard linear systems techniques, we can readily solve (2) in the transform domain to obtain

$$\hat{\mathbf{P}}(z;t) = \exp\left(\left[-\mathbf{\Lambda} + z\mathbf{Q}\mathbf{B}^{\mathrm{T}}\right]t\right)\hat{\mathbf{P}}(z;0), \quad (3)$$

where $\hat{\mathbf{P}}(z;t)$ is the z-transform of $\mathbf{P}_i(t)$, defined as

$$\hat{\mathbf{P}}(z;t) = \sum_{i=0}^{\infty} z^i \mathbf{P}_i(t).$$

As an immediate application, (3) yields important statistics of the fractal counting process, which are of fundamental interest in a broad range of scenarios such as optimal buffer allocation for queueing systems, queueing delay estimation, and shot-noise analysis. In particular, through transform inversion, we can obtain the counting process distribution, $\pi_i(t)$, which gives the probability of i arrivals in the time interval $(0,t]$. With the time of reference $t = 0$ chosen at a renewal time, we can analyze this distribution as perceived by an arrival of the process. In the context of queueing, for example, this could represent the customer arrival process seen by a customer, and is thus essential in waiting time calculation. For this case, the scale of the first interarrival is chosen according to the probabilities in $\mathbf{Q}$, and the initial conditions in (3) are thus $\hat{\mathbf{P}}(z;0) = \mathbf{Q}$. This leads to the closed-form result

$$\begin{aligned}\pi_0(t) &= \mathbf{1}^{\mathrm{T}}\hat{\mathbf{P}}(0;t) \\ &= \sum_{i=1}^{L} \sigma^2 q^{i-1} \exp\left(-\frac{\lambda}{\eta^{i-1}}t\right),\end{aligned}$$

where $\mathbf{1}$ is a vector of all 1's. More generally, other terms of the counting process distribution can be obtained numerically. Fig. 2 shows the first four terms of the counting process distribution of a fractal renewal process with $\gamma = 1.8$, computed with a dyadic (i.e., $\eta = 2$) multiscale representation. Many key properties of fractal point processes can be inferred from these plots. For example, the first few terms of the distribution peak near the origin, agreeing with the strong tendency of clustering in a fractal point process. At the other extreme, the decay in the distribution approaches $t^{\gamma-1}$ for large values of t, which reflects the relation between the counting process distribution and the running integral of the interarrival density. Moreover, this heavy-tail polynomial decay is in close agreement with the long quiescent gaps typically found between clusters.

In many applications, the counting process distribution as viewed upon random incidence is equally important. In queueing theory, for example, random incidence is extremely useful for portraying customer arrivals as seen by a queueing system, and is thus important in optimal server and buffer allocation. Under this mode of observation, the time of reference is independent of the counting process, and the scale of the first interarrival is chosen according to the probabilities in $\lim_{t\to\infty}\hat{\mathbf{P}}(1;t)$, the steady-state scale distribution. It can be shown [8] that this limit exists in general, and that it is $(\hat{\sigma}^2, \hat{\sigma}^2 2q, \ldots, \hat{\sigma}^2(2q)^{L-1})^{\mathrm{T}}$, where $\hat{\sigma}^2$ is a normalization factor. Using this set of the initial conditions and taking the same approach as before, we can compute from (3) the counting process distribution viewed upon random incidence. We remark that one of the most notable features of the resulting distribution is the dominance by the probability of zero arrivals, which again reflects the tremendous spacing between clusters.

3. RANDOM ERASURE OF FRACTAL RENEWAL PROCESSES

A point process transformation of general importance is random erasure, or random removal of events. In networking, for example, random erasure provides a natural model for the branching of packet streams, as well as packet loss due to corruption. As a preliminary study, we consider the effects of Bernoulli erasure on fractal renewal processes, whereby deletion of each point occurs independently with the same probability p. This mode of erasure is a realistic model for many scenarios, and is also highly tractable.

Under Bernoulli erasure, an arrival of the original process in general contributes a count of unity to the erased process with probability $1-p$, and zero with probability p. This leads to the convenient transform-domain relation

$$\hat{\pi}_e(z;t) = \hat{\pi}\left(p + (1-p)z;t\right), \quad (4)$$

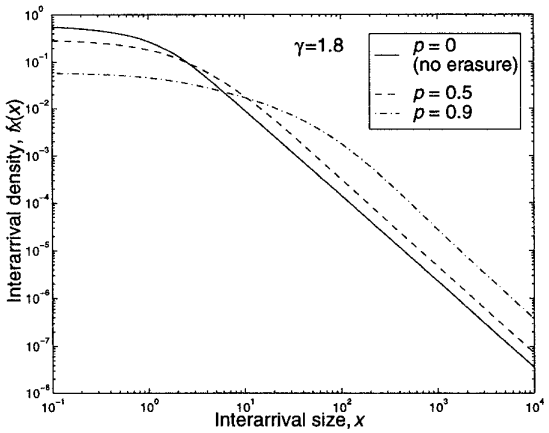

Figure 3: *Interarrival density for a fractal renewal process under Bernoulli erasure.*

where $\hat{\pi}_e(z;t)$ and $\hat{\pi}(z;t)$ are the z-transforms of the erased and original counting processes, respectively. Setting $z = 0$, we immediately obtain from (4) the probability of 0 arrivals of the erased process in the interval $(0, x]$, conditioned on an arrival at $t = 0$. In turn, this event is equivalent to the interarrival X being greater than or equal to x. Thus, using (3) in (4), and setting $z = 0$, we obtain for an erased fractal renewal process

$$\Pr\{X \geq x\} = \mathbf{1}^{\mathrm{T}} \exp\left(\left[-\mathbf{\Lambda} + p\mathbf{Q}^{\mathrm{T}}\mathbf{B}\right] x\right) \mathbf{Q}$$

which, upon differentiation, leads to the interarrival density

$$f_X(x) = \mathbf{1}^{\mathrm{T}}(\mathbf{\Lambda} - p\mathbf{Q}^{\mathrm{T}}\mathbf{B}) \exp\left(\left[-\mathbf{\Lambda} + p\mathbf{Q}^{\mathrm{T}}\mathbf{B}\right] x\right) \mathbf{Q}. \quad (5)$$

Using (5), we have plotted in Fig. 3 the interarrival density of a fractal renewal process with shape parameter $\gamma = 1.8$, subject to various erasure probabilities. In general, while Bernoulli erasure reduces the number of short interarrivals, these plots suggest that on coarse scales, the self-similar structure is largely preserved.

4. SUPERPOSITION OF FRACTAL RENEWAL PROCESSES

In many situations, overall effects of coexisting point processes take precedence over the individual constituents. As a prime example, networking design is typically concerned with the aggregate usage of multiple users, rather than any individual. More generally, point process superposition also arises naturally in many other contexts, including overall distribution of coexisting man-made and natural terrain features, and population distribution of a collection of species. While it is well known that the Poisson family constitutes a domain of attraction for superposition, self-similarity in network traffic data suggests similar qualities of the class of fractal point processes. As a preliminary study of this attraction behavior, we explore a somewhat simplified but illuminating problem, the closure of fractal renewal processes under superposition.

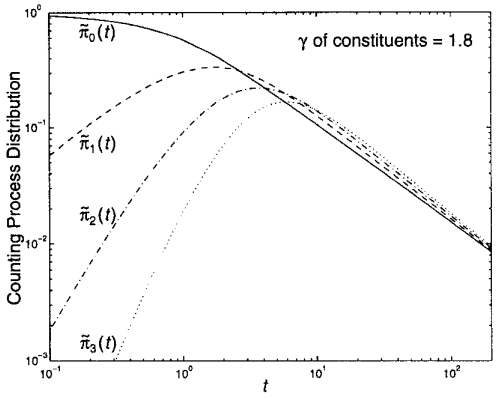

Figure 4: *Counting process distribution of the superposition of two independent fractal renewal processes.*

While analysis of fractal point process superposition is traditionally cumbersome, insightful results can be obtained via our multiscale framework. In general, for independent constituents with counting process distributions $\pi_i^{(1)}(t)$ and $\pi_i^{(2)}(t)$, the counting process distribution of their superposition, denoted as $\tilde{\pi}_i(t)$, is obtained via a discrete convolution:

$$\tilde{\pi}_i(t) = \sum_{j=0}^{i} \pi_j^{(1)}(t)\pi_{i-j}^{(2)}(t). \quad (6)$$

Thus, exploiting our results of Section 2, we can obtain a counting process characterization of the superposition of independent fractal renewal processes, as observed by an arrival of one of the constituents or upon random incidence [8]. Fig. 4 shows an arrival-observed counting process distribution corresponding to the superposition of two independent fractal renewal processes with shape parameter $\gamma = 1.8$, computed with via (6). Comparing this set of plots with those for a single process (Fig. 2), we see that key features such as the asymptotic power-law decay, are preserved under superposition, suggesting invariance of fractal renewal processes under this transformation.

5. QUEUEING PROBLEMS INVOLVING FRACTAL RENEWAL PROCESSES

Studies in queueing theory are aimed primarily at the assessment and design of scheduling and allocation schemes for shared resources, ranging from various computing and manufacturing facilities to human resources in a variety of service industries. In telephony applications, the subject has also generated particularly useful models for networking delay. In fact, together with random erasure and superposition, queueing adequately captures the activities in a broad class of networks. While various queueing problems involving Poisson arrivals have been adequately solved (see, e.g., [7]), discrepancy between real traffic data and the Poisson model raises questions regarding the appropriateness of these results in practice. As a result, design of networking protocols such as flow control and routing has remained

largely ad hoc. Using our multiscale framework, we consider in this section queueing problems involving fractal point processes, which better match traffic data in a broad class of communication networks (see, e.g., [4],[5]).

The basis of our development is the multiscale birth-and-death model, which is a powerful generalization of the well-known birth-and-death model [7]. For simplicity, we focus this preliminary analysis on queueing systems with single memoryless servers driven by fractal renewal process input; the idea for more general situations is similar. The state space of the multiscale birth-and-death process is identical to the multiscale pure-birth process, with the superstates now representing the number of customers in the system. To model service completion and customer departure, downward transitions, or deaths, are added. Since bulk service is not considered, each death decreases the customer count by 1. Moreover, since service is assumed independent of the arrival process, a death does not result in a scale change. Thus, we add to the multiscale pure-birth process transitions of the form $(i,j) \to (i-1,j)$ for every $i > 0$, with rates all equal to the service rate μ. As an interesting observation, we remark that reversing the roles of birth and death transitions results in a dual process which is particularly important for modeling other telecommunication queueing systems, such as those involving heavy-tailed holding time.

In our analysis, the service rate is assumed sufficiently high to allow for a steady-state solution. Also, the probability of an arbitrarily long waiting line is assumed negligible, so that truncation of the state space at some superstate N is justified. Under these assumptions, we have a system of steady-state equations

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{\Gamma P}_0 && \text{(7a)} \\ \mathbf{P}_i &= (\mathbf{\Gamma}+\mathbf{I})\mathbf{P}_{i-1} - \mathbf{C}\mathbf{P}_{i-2}, \quad 2 \le i \le N && \text{(7b)} \\ \mathbf{0} &= \mathbf{P}_N - \mathbf{C}\mathbf{P}_{N-1}, && \text{(7c)} \end{aligned}$$

where $\mathbf{P}_i$ are the steady-state probabilities $\lim_{t\to\infty} \mathbf{P}_i(t)$, $\mathbf{\Gamma} \triangleq \mu^{-1}\mathbf{\Lambda}$, and $\mathbf{C} \triangleq \mu^{-1}\mathbf{Q}\mathbf{B}^{\mathrm{T}}$. It can be shown [8] that to solve (7), it suffices to find $\mathbf{P}_i$ such that

$$\begin{aligned} \mathbf{P}_0 &\in \mathrm{Null}(\mathbf{M}_N) \\ \mathbf{P}_i &= \mathbf{A}_i\mathbf{P}_0, \end{aligned}$$

where the matrices $\mathbf{A}_i$ and $\mathbf{M}_i$ are governed by the recurrence relation

$$\begin{bmatrix} \mathbf{A}_i \\ \mathbf{M}_i \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{I} \\ \mathbf{\Gamma}-\mathbf{C} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{i-1} \\ \mathbf{M}_{i-1} \end{bmatrix}, \quad 1 \le i \le N,$$

with the initial conditions $\mathbf{A}_0 = \mathbf{I}$ and $\mathbf{M}_0 = \mathbf{0}$.

Using the above, we have obtained and plotted in Fig. 5 the steady-state customer distribution for a queueing system driven by a fractal renewal process input with $\gamma = 1.8$. The arrival and service rates are related via $\mu = .8\lambda.$, and the state space (i.e., queue length) was truncated at $N = 20$. As a consequence of input clustering, the distribution exhibits a dichotomy: on one hand, the idle probability is exceptionally high, owing to the long quiescent periods between customer clusters, while on the other, the system behaves like an M/M/1 queue when servicing clusters, as manifested by the geometric progression in distribution (see *e.g.*, [7], Ch. 3).

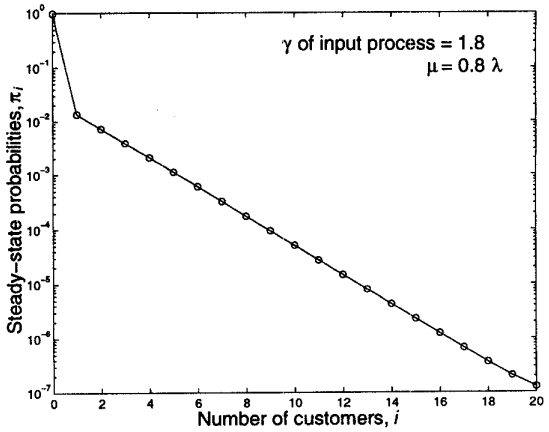

Figure 5: *Steady-state customer distribution in a queueing system with a memoryless server and fractal renewal process input.*

In conclusion, the results of this paper suggest that our multiscale framework leads to tractable analysis of fractal point processes under important transformations for engineering applications. A more extensive development of these results, along with the corresponding network design issues, is contained in [8].

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