Correspondence

A Karhunen–Loève-like Expansion for $1/f$ Processes via Wavelets

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Abstract—**While so-called $1/f$ or scaling processes emerge regularly in modeling a wide range of natural phenomena, as yet no entirely satisfactory framework has been described for the analysis of such processes. Orthonormal wavelet bases are used to provide a new construction for nearly $1/f$ processes from a set of uncorrelated random variables.**

I. Introduction

There are many physical phenomena for which measured spectra are roughly of the form

$$S(\omega)\propto\frac{1}{|\omega|^{\gamma}} \tag{1}$$

over several frequency decades, where γ is some parameter in the range $0<\gamma<2$ [3]. While these do not constitute valid power spectra in the theory of stationary processes, a variety of attempts have been made to explain such spectra through nonstationary processes and notions of generalized spectra [2], [3], [5], [6]. However, as yet no universal framework for characterizing and analyzing $1/f$ processes has been found. In this correspondence, we construct nonstationary processes from orthonormal wavelet expansions in terms of uncorrelated random variables and show how to extend the concept of spectra for these processes in a manner consistent with measured spectra. Via this construction we then obtain processes having an extended spectrum of the general form

$$\frac{k_1}{|\omega|^{\gamma}}\le S(\omega)\le\frac{k_2}{|\omega|^{\gamma}} \tag{2}$$

for some $0<k_1\le k_2<\infty$. These may be considered *generalized* or *nearly* $1/f$ processes, for they retain the basic macroscopic spectral structure usually associated with $1/f$ phenomena.

II. Results from Wavelet Theory

We present some results required from the theory of orthonormal wavelet expansions. Comprehensive treatments can be found in Mallat [4] and Daubechies [1].

Orthonormal wavelet expansions of $L^2(\mathbf{R})$ functions employ real orthonormal basis functions of the form

$$\psi_n^m(t)=2^{m/2}\psi(2^m t-n) \tag{3}$$

where $\psi(t)$ is the basic wavelet and has a band-pass Fourier transform $\Psi(\omega)$. These expansions have the special property that all the basis functions are dilations and translations of each other, and m,n are the integer dilation and translation indices, respectively.

Such representations arise rather naturally out of a theory of multiresolution analysis. A resolution-limited representation of a function $x(t)$ that discards details on scales smaller than 2^M is obtained from the partial expansion

$$x^M(t)=\sum_{m<M}\sum_n d_n^m\psi_n^m(t) \tag{4}$$

where the coefficients d_n^m are obtained via projection. In turn, $x^m(t)$ can be decomposed into the orthonormal expansion

$$x^m(t)=\sum_n c_n^m\phi_n^m(t) \tag{5}$$

where the $\phi_n^m(t)$ are also related by dilations and translations:

$$\phi_n^m(t)=2^{m/2}\phi(2^m t-n). \tag{6}$$

The scaling function $\phi(t)$ has a low-pass Fourier transform $\Phi(\omega)$ satisfying

$$|\Phi(\omega)|\le 1,\qquad |\Phi(0)|=1. \tag{7}$$

Note that, for a given m, the c_n^m capture the information in the signal at resolution 2^m, and the d_n^m capture the new information or *detail* in the signal going from resolution 2^m to resolution 2^{m+1}. It is possible to construct these multiresolution representations so that the information sequences are related through filter-downsample and upsample-filter relations:

$$c_n^m=\sum_k h[k-2n]c_k^{m+1}, \tag{8}$$

$$d_n^m=\sum_k g[k-2n]c_k^{m+1}, \tag{9}$$

$$c_n^{m+1}=\sum_k\{h[n-2k]c_k^m+g[n-2k]d_k^m\}, \tag{10}$$

where $h[n]$ and $g[n]$ are appropriately defined *conjugate quadrature* discrete-time filters having Fourier transforms $H(\omega)$ and $G(\omega)$, respectively. These filters satisfy, among other properties,

$$G(\omega)=e^{-j\omega}H^*(\omega+\pi) \tag{11}$$

and

$$|H(\omega)|^2+|H(\omega+\pi)|^2=1. \tag{12}$$

With such a formulation, the scaling function and basic wavelet are related through

$$\Phi(\omega)=H(\omega/2)\Phi(\omega/2), \tag{13}$$

$$\Psi(\omega)=G(\omega/2)\Phi(\omega/2). \tag{14}$$

III. $1/f$ Processes from Wavelets

Consider the construction of a random process $x(t)$ from an orthonormal wavelet basis. Define

$$x_M(t)=\sum_{m\ge M}\sum_n d_n^m\psi_n^m(t) \tag{15}$$

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as the resolution-limited approximation to $x(t)$ for which information at resolutions lower than 2^M is discarded, so

$$x(t) = \lim_{M \to -\infty} x_M(t) = \sum_{m,n} d_n^m \psi_n^m(t). \tag{16}$$

Suppose that for arbitrary distinct pairs m and m', the detail sequences d_n^m and $d_n^{m'}$ are uncorrelated, and suppose that for each m, d_n^m is wide-sense stationary with spectrum $P_m(\omega)$. Then $x_M(t)$ is cyclostationary [7] with period 2^{-M} and has the associated time-averaged spectrum

$$S_M(\omega) = \sum_{m \geq M} P_m(\omega) |\Psi(2^{-m}\omega)|^2. \tag{17}$$

Since any attempt to measure a spectrum for an $x(t)$ generated by this process involves the use of finite-length data records, information at lower resolutions is invariably lost. Hence, except at the lower frequencies, (17) is a reasonable representation of a measured spectrum for $x(t)$. We can therefore define a limiting spectrum for the nonstationary $x(t)$ through

$$S(\omega) = \lim_{M \to -\infty} S_M(\omega) = \sum_m P_m(\omega) |\Psi(2^{-m}\omega)|^2 \tag{18}$$

that is consistent with a measured spectrum corresponding to arbitrarily long data records.

Next, let us consider the case where for some $0 < \gamma < 2$ and for each m the sequences d_n^m are white with

$$P_m(\omega) = 2^{-\gamma m}\sigma^2. \tag{19}$$

Let us further set $\sigma^2 = 1$ without any loss of generality. By substituting (19) into (18) and applying, in order, (14), (11), (12), and (13), we readily obtain

$$S(\omega) = (2^\gamma - 1) \sum_m 2^{-\gamma m} |\Phi(2^{-m}\omega)|^2 \tag{20}$$

where for $\omega \neq 0$ the summation is convergent.

Finally, from (20) we can show that there exist $0 < k_1 \leq k_2 < \infty$ such that (2) is valid, i.e., that $x(t)$ is nearly $1/f$, for any wavelet basis for which $\Phi(\omega)$ is continuous at $\omega = 0$ and $|\Phi(\omega)|$ decays at least as fast as $1/\omega$.

Before proceeding to a proof of this result, it is worth remarking that most reasonable wavelet bases satisfy these requirements. These include Daubechies' class of compactly supported wavelets [1], and the class of Battle–Lemarie wavelets derived by orthogonalizing Nth order spline functions [1]. For the latter class of wavelets, it has been shown that $\Psi(\omega)$ decays like $1/\omega^N$. Special cases include the well-known Haar-wavelet basis for $N = 1$ that has

$$\phi(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t < 0, t > 1 \end{cases} \tag{21}$$

and the sinc-wavelet basis for $N \to \infty$ that has

$$\Phi(\omega) = \begin{cases} 1, & |\omega| \leq \pi \\ 0, & |\omega| > \pi. \end{cases} \tag{22}$$

It is possible to establish from simple geometric arguments that the best bounding constants for the sinc-wavelet case are

$$k_1 = \pi^\gamma, \tag{23}$$

$$k_2 = (2\pi)^\gamma. \tag{24}$$

It is also possible to attach a special interpretation to the case $\gamma = 1$, arguably the most prevalent of the $1/f$ processes. In this case, the choice of $P_m(\omega) = 2^{-m}$ corresponds to distributing power equally among the detail signals at all resolution scales, since we have for each m

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} P_m(\omega) |\Psi(2^{-m}\omega)|^2 \, d\omega = 1. \tag{25}$$

We turn now to a proof of our main result. From (20) we get, for any n,

$$S(\omega) = 2^{-n\gamma} S(2^{-n}\omega). \tag{26}$$

Moreover, given ω we can choose m_0 and ω_0 such that $\omega = 2^{m_0}\omega_0$ and $1 \leq |\omega_0| < 2$. Hence, from $S(\omega) = 2^{-m_0\gamma} S(\omega_0)$ it follows that

$$\left[\inf_{1 \leq |\omega_0| < 2} S(\omega_0) \right] |\omega|^{-\gamma} \leq S(\omega) \leq \left[\sup_{1 \leq |\omega_0| < 2} S(\omega_0) \right] 2^\gamma |\omega|^{-\gamma}. \tag{27}$$

It suffices, therefore, to find upper and lower bounds for $S(\omega_0)$ on $1 \leq |\omega_0| < 2$.

Since $\Phi(\omega)$ decays at least as fast as $1/\omega$ and is bounded, there exists a $C \geq 1$ such that

$$|\Phi(\omega)| \leq \frac{C}{1 + |\omega|}. \tag{28}$$

Using this with (7) in (20) leads to the upper bound:

$$S(\omega_0) \leq (2^\gamma - 1) \left[\sum_{m=0}^{\infty} 2^{-\gamma m} + \sum_{m=1}^{\infty} 2^{\gamma m} C^2 2^{-2m} \right] < \infty. \tag{29}$$

Since Φ is continuous at 0 and $|\Phi(0)| = 1$, there exists n_0 such that $|\Phi(\omega)| > 1/2$ when $|\omega| < 2^{-n_0}$. Hence,

$$|\Phi(2^{-n_0-1}\omega_0)| > 1/2 \tag{30}$$

from which the lower bound

$$S(\omega_0) \geq (2^\gamma - 1) 2^{-\gamma(n_0+1)} |\Phi(2^{-n_0-1}\omega_0)|^2$$
$$\geq (2^\gamma - 1) 2^{-\gamma(n_0+1)-2} > 0 \tag{31}$$

follows. □

As a final remark, it should be noted that the spectrum constructed in (18) corresponds to an infinite-variance process, consistent with the fact that processes with spectra of the form (1) have infinite-variance. For the case $0 < \gamma \leq 1$ the problem arises in the tails of the spectrum, while for the case $1 \leq \gamma < 2$ the problem arises in a neighborhood of the spectral origin. With slight modifications to the constructions of this correspondence, it is possible to be more careful in the treatment of these issues. In particular, recognizing that it is mainly the low-frequency behavior of $1/f$ processes that is of interest, we may omit the finer scales in the construction of (15), and thereby avoid the problem in the spectral tails. We are then assured that (17) is a valid spectrum, and, provided $0 < \gamma < 1$, that the limiting spectrum (18) is valid as well. For $1 \leq \gamma < 2$, however, it is impossible to avoid the limit process having infinite variance.

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Variable-to-Fixed Length Codes are Better than Fixed-to-Variable Length Codes for Markov Sources

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***Abstract*—It is demonstrated that for finite-alphabet, Kth order ergodic Markov Sources (i.e., memory of K letters), a variable-to-fixed code is better than the best fixed-to-variable code (Huffman code). It is shown how to construct a variable-to-fixed length code for a Kth order ergodic Markov source, which compresses more effectively than the best fixed-to-variable code (Huffman code).**

I. Introduction

Consider the class of finite-alphabet, finite-order ergodic Markov sources, characterized by a probability distribution of the form

$$P(X)=\prod_{i=1}^{n}P(X_i|X_1^{i-1}) \tag{1}$$

where

$$X=X_1,X_2,\cdots,X_n$$

$$P(X_i|X_1^{i-1})=P(X_i|X_{i-K}^{i-1}),\qquad \text{for any } i\geq K,$$

and where

1) $X_i^j \triangleq X_i,X_{i+1},\cdots,X_j, i<j$,
2) X_i is the output of the source at the ith instant, $X_i\in A$; $|A|=\alpha$.

A *code* is an extended alphabet C of M vectors ("words") $X_1^c,X_2^c,\cdots,X_M^c$ where $X_i^c\in A^{l(i)}$ and where $l(i)$ is the length of the vector X_i^c.

Assume also that any vector $x\in A^l$ for $l\geq \max_i l(i)$ has a prefix $X_i^c\in C$ for some $1\leq i\leq M$, and that for every i and $j(i\neq j)X_i^c\in C$ is not a prefix of $X_j^c\in C$ (i.e., the code is complete and proper [1]).

Every vector $X_i^c\in C$ is mapped into a unique binary sequence Y_i^c of length $L(Y_i^c)\triangleq L(i)$ binary letters. This sequence is called the "codeword" for the word X_i^c. A fixed-to-variable length code (FVL) is one for which

$$l(i)=l,\qquad 1\leq i\leq M. \tag{2}$$

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A variable-to-fixed length code (VFL) is one for which

$$L(i)=L,\qquad 1\leq i\leq M;$$
$$L=\lceil \log M\rceil \tag{3}$$

where logarithms in this correspondence are taken to be of base 2.

Consider the parsing of $X=X_1^n$ into a sequence of $q_c(X)$ words of C (ignoring end-effects)

$$X=X^1,X^2,X^3,\cdots,X^j,\cdots,X^{q_c(X)}, \tag{4}$$

where $X^j\in C$, $1\leq j\leq q_c(X)$. Let

$$L_c(X)=\sum_{j=1}^{q_c(X)}L(j), \tag{5}$$

where $L(j)$ is the length of Y^j, the binary codeword that corresponds to the jth word in the parsed X. The compression-ratio for a given code C is defined by

$$\rho_c=\lim_{n\to\infty}\frac{EL_c(X)}{n\log\alpha}, \tag{6}$$

where $E(\cdot)$ denotes expectation.

It is well known [2] that

$$\rho_c\geq\frac{H}{\log\alpha} \tag{7}$$

and that there exist a sequence of FVL codes (Huffman codes) such that

$$\lim_{M\to\infty}\rho_c=\frac{H}{\log\alpha}\triangleq\rho(\infty) \tag{8}$$

where

$$H=\lim_{n\to\infty}-\frac{1}{n}E\log P(X). \tag{9}$$

Unfortunately, for finite-order Markov sources with memory ($K>1$) and with $\rho(\infty)<1$ we have that

$$\rho(M)\triangleq\min\rho_c>\rho(\infty) \tag{10}$$

where the minimization is carried over *all* codes with M code-words.

In Theorem 1, we derive lower-bounds on ρ_c, for any code such that the shortest word in C is no shorter than K. Clearly, any FVL code with more than α^K codewords is included in this family of codes.

In Theorem 2, we derive upper bounds on ρ_c for a VFL code and show that it approaches the lower bound of Theorem 1, at least for sources with large memory ($K\gg 1$).

At the same time, the rate of approach of ρ_c for the best FVL code (i.e., Huffman code) is *slower* than that of the VFL code. Thus, VFL coding takes better advantage of the source memory.

II. Derivations and Statement of Results

The coding of a sequence X was shown to be associated with parsing the sequence into $q_c(X)$ words.

Each word is encoded into one out of M codewords of the given code C. The selection of the particular codeword is independent of the past words, without taking advantage of the memory of the source. Thus, when encoding each of the $q_c(X)$ words in X, there is a certain loss in compression. We show that the accumulated average loss for X is proportional to the expected number of words $Eq_c(X)$, and demonstrate that

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