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SYNTHESIZING SELF-SYNCHRONIZING CHAOTIC ARRAYS

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A systematic approach is developed for synthesizing dissipative chaotic arrays that possess the self-synchronization property. The ability to synthesize high-dimensional chaotic arrays further enhances the usefulness of synchronized chaotic systems for communications, signal processing, and modeling of physical processes.

1. Introduction

Self-synchronization of chaotic systems is an intriguing concept, and recently, has received considerable attention. This property allows two identical chaotic systems to synchronize when the second system (receiver) is driven by the first (transmitter) [Pecora & Carroll, 1990 & 1991; Carroll & Pecora, 1991]. The ability to synchronize remote chaotic systems by linking them with a common drive signal(s) suggests new and potentially useful approaches to private communications [Oppenheim *et al.*, 1992; Kocarev *et al.*, 1992; Parlitz *et al.*, 1992; Halle *et al.*, 1993; Cuomo *et al.*, 1993].

To further enhance the applicability of synchronized chaotic systems for communications, a systematic approach was needed for creating these types of systems. This issue was addressed in Cuomo [1993a], where Lyapunov's direct method was used to develop a systematic procedure for synthesizing a class of high-dimensional dissipative chaotic systems that possess the self-synchronization property. While those systems appear to be very promising, they seem to exhibit only a single positive Lyapunov exponent. This limitation imposes a constraint on the complexity of the chaotic dynamics, which may be undesirable in certain private communication applications.

The potential for creating more complex chaotic systems motivates our current study of mutually coupled chaotic systems. Such systems are of great interest and have been studied extensively. Kowalski *et al.* [1990], for example, has investigated the complex interactions among as ensemble of mean-field coupled Lorenz oscillators [Lorenz, 1963]. Complex chaotic behavior has also been investigated in arrays of Rössler oscillators [Waller & Kapral, 1984; Klevecz *et al.*, 1992], laser systems [Winful & Rahman, 1990], neural networks [Hansel & Sompolinsky, 1992], Selkov models [Badola *et al.*, 1991], and electronic circuits [Rulkov *et al.*, 1992]. Although significant progress has been made toward understanding these systems, systematic synthesis procedures for self-synchronizing chaotic arrays have not been developed.

In this paper, we utilize Lyapunov functions to develop a systematic synthesis capability for a class of chaotic arrays which possess the self-synchronization property. These arrays offer considerable flexibility in the design of complex chaotic systems; they may contain an unlimited number of Lorenz oscillators and an N-dimensional linear system. The linear system provides for both integrated and direct coupling between each Lorenz oscillator. The advantages of linearly coupling several Lorenz

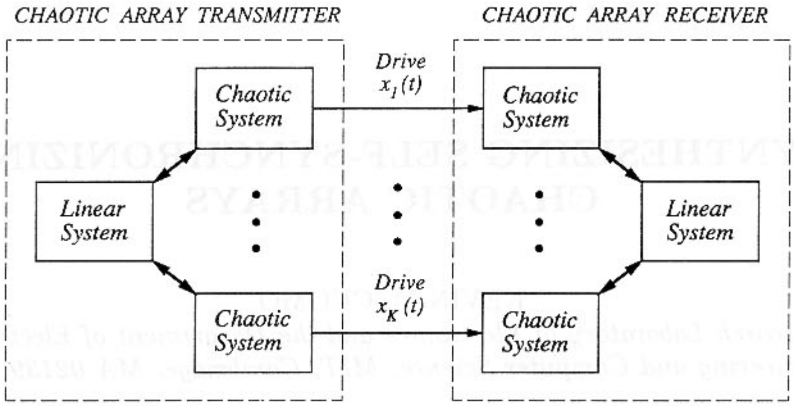

Fig. 1. Communicating with chaotic arrays.

oscillators are that the resulting chaotic arrays are analytically tractable and have a modular structure which makes them straightforward to implement.

Figure 1 illustrates a communication scenario in which the transmitter array conveys a *set* of drive signals to an identical receiver array. These drive signals provide a means for establishing and maintaining synchronization between the transmitter and receiver arrays. Although the transmitter and receiver arrays can exhibit very complex dynamics, they will be completely synchronized if certain conditions, to be determined later, are satisfied. A potential drawback of this approach for communication applications is that the synchronization requires that more than one drive signal be communicated — one drive signal for each Lorenz oscillator. This requirement increases the complexity of the communication system. There are, however, potential advantages to this approach. The utilization of several drive signals could make it more difficult for an unintended listener to obtain synchronization with the transmitter. Chaotic arrays are also highly modular and easy to modify. Increasing the complexity of the chaotic dynamics can be achieved by simply adding additional oscillators to the transmitter and receiver.

Our first goal in this paper is to develop sufficient conditions for which the transmitter array satisfies two requirements: (i) there exists an algebraically similar receiver system which possesses the global self-synchronization property, and (ii) the transmitter system is globally stable. These requirements are satisfied in Secs. 2.1 and 2.2, respectively. In Sec. 2.3, we summarize the various self-synchronization and global stability conditions for this class of systems and suggest a systematic procedure for synthesizing chaotic arrays. In Sec. 3, we design a low-order chaotic array and demonstrate its nonlinear dynamical behavior with several numerical experiments. Section 4 summarizes the main results of the paper.

2. Theory

There are several ways to linearly couple a set of Lorenz oscillators. With our approach, the chaotic signals $z(t)$ from each Lorenz oscillator drive the linear system, and the resultant outputs are added to the appropriate oscillator's equation for $\dot{z}$. This type of array can be represented by a set of state equations of the form

$$\begin{aligned} \dot{x}_i &= \sigma_i(y_i - x_i)\,, \\ \dot{y}_i &= r_i x_i - y_i - x_i z_i\,, \\ \dot{z}_i &= x_i y_i - b_i z_i + o_i\,, \\ \dot{\mathbf{l}} &= A\mathbf{l} + B\mathbf{z}\,, \\ \mathbf{o} &= C\mathbf{l} + D\mathbf{z}\,, \end{aligned} \tag{1}$$

where the subscript $i = 1, \ldots, K$ denotes the individual Lorenz oscillators. The vectors

$$\mathbf{l} = \begin{bmatrix} l_1 \\ \vdots \\ l_N \end{bmatrix},\ \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_K \end{bmatrix},\ \mathbf{o} = \begin{bmatrix} o_1 \\ \vdots \\ o_K \end{bmatrix},$$

denote the state variables, inputs, and outputs of the linear system, respectively.

In (1), the linear system is N-dimensional with K inputs and K outputs. Therefore, the matrices A, B, C, and D have dimension $N \times N$, $N \times K$, $K \times N$, and $K \times K$ respectively. For notational simplicity, we will denote the state variables in (1) collectively by the vector $\mathbf{v} = (x_1, y_1, z_1, \ldots, x_K, y_K, z_K, \mathbf{l})$ when convenient. Below, we determine constraints on the free parameters of the transmitter array which guarantees that it possesses the global self-synchronization property.

2.1. *Conditions for global self-synchronization*

From certain theoretical and practical viewpoints, it is advantageous for the receiver array to have the same algebraic structure as the transmitter array. The self-synchronization properties of the Lorenz system, discussed in Cuomo & Oppenheim [1993], suggest a receiver system of the form

$$\begin{aligned}
\dot{x}_{ir} &= \sigma_i(y_{ir} - x_{ir})\,, \\
\dot{y}_{ir} &= r_i x_i(t) - y_{ir} - x_i(t) z_{ir}\,, \\
\dot{z}_{ir} &= x_i(t) y_{ir} - b_i z_{ir} + o_{ir}\,, \\
\dot{\mathbf{l}}_r &= A\mathbf{l}_r + B\mathbf{z}_r\,, \\
\mathbf{o}_r &= C\mathbf{l}_r + D\mathbf{z}_r\,.
\end{aligned} \tag{2}$$

Algebraically, the receiver system (2) is obtained from the transmitter (1) by renaming variables $\mathbf{v} \to \mathbf{v}_r$ and substituting the drive signals $x_i(t)$ for $x_{ir}(t)$ in the equations for $\dot{y}_{ir}$ and $\dot{z}_{ir}$.

We can study the self-synchronization properties of the transmitter and receiver arrays by forming the error system. The error system is derived by defining the error variables

$$\begin{aligned}
e_{xi}(t) &= x_i(t) - x_{ir}(t)\,, \\
e_{yi}(t) &= y_i(t) - y_{ir}(t)\,, \\
e_{zi}(t) &= z_i(t) - z_{ir}(t)\,, \\
\mathbf{e}_l(t) &= \mathbf{l}(t) - \mathbf{l}_r(t)\,,
\end{aligned}$$

and subtracting (2) from (1) to obtain

$$\begin{aligned}
\dot{e}_{xi} &= \sigma_i(e_{yi} - e_{xi})\,, \\
\dot{e}_{yi} &= -e_{yi} - x_i(t) e_{zi}\,, \\
\dot{e}_{zi} &= x_i(t) e_{yi} - b_i e_{zi} + C_i \mathbf{e}_l + D_i \mathbf{e}_z\,, \\
\dot{\mathbf{e}}_l &= A\mathbf{e}_l + B\mathbf{e}_z\,.
\end{aligned} \tag{3}$$

In (3), we denote the K rows of C by C_i and the K rows of D by D_i. The error vector $\mathbf{e}_z$ denotes the K error variables corresponding to $e_{zi}(t)$, *i.e.*, $\mathbf{e}_z = (e_{z1}, \ldots, e_{zK})$.

A set of sufficient conditions for the error system to be globally asymtotically stable at the origin can be derived by considering a Lyapunov function of the form

$$E = \frac{1}{2}\left(\sum_{i=1}^{K}\left(\frac{1}{\sigma_i}e_{xi}^2 + e_{yi}^2 + e_{zi}^2\right) + \mathbf{e}_l^T R \mathbf{e}_l\right),$$

where R is a symmetric $N \times N$ positive definite matrix. The time rate of change of E along trajectories is given by

$$\begin{aligned}
\dot{E} = \sum_{i=1}^{K}\left\{\begin{bmatrix} e_{xi} \\ e_{yi}\end{bmatrix}^T \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} e_{xi} \\ e_{yi}\end{bmatrix}\right\} \\
- \begin{bmatrix} \mathbf{e}_z \\ \mathbf{e}_l \end{bmatrix}^T T \begin{bmatrix} \mathbf{e}_z \\ \mathbf{e}_l \end{bmatrix},
\end{aligned} \tag{4}$$

where the matrix T is given by

$$T = \begin{bmatrix} \Lambda_b - \frac{1}{2}(D + D^T) & -\frac{1}{2}(B^T R + C) \\ -\frac{1}{2}(RB + C^T) & -\frac{1}{2}(RA + A^T R) \end{bmatrix}.$$

Notice that T contains the diagonal matrix $\Lambda_b = \operatorname{diag}(b_1, \ldots, b_K)$. The diagonal elements of Λ_b correspond to the set of b parameters for the ensemble of Lorenz oscillators.

Observe that $\dot{E}$ is negative definite if T is positive definite. A sufficient set of conditions for T to be positive definite are given below.

- $RB + C^T = 0$, for some $N \times N$ symmetric positive definite matrix R.
- $RA + A^T R$ is negative definite.
- $\Lambda_b - \frac{1}{2}(D + D^T)$ is positive definite.

The first condition provides a constraint between the allowable B and C matrices. The second condition can always be satisfied if A is a stable matrix.[1] The third condition provides a bound on D. If these conditions are satisfied, then the transmitter and receiver arrays are guaranteed to synchronize regardless of their initial conditions. It should also be noted that these conditions are not unique. We have chosen the obvious conditions that block diagonalize T into positive definite blocks; however, there are many other possibilities.

[1] The term "stable matrix" used here refers to a matrix having all of its eigenvalues in the left-half plane.

2.2. *Conditions for global stability*

A set of sufficient conditions for which all trajectories of the transmitter equations remain bounded can be determined by defining a family of ellipsoids

$$V(\mathbf{v}) = \frac{1}{2}(\mathbf{x}^T\Lambda_r\mathbf{x} + \mathbf{y}^T\Lambda_\sigma\mathbf{y} + (\mathbf{z}-2\mathbf{r})^T\Lambda_\sigma(\mathbf{z}-2\mathbf{r}) + \mathbf{l}^T P\mathbf{l}) = k\,, \quad (5)$$

where P is a symmetric $N \times N$ positive definite matrix and k is a positive scalar. The vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in (5) denote the state variables of the Lorenz oscillators and are given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_K \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_K \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_K \end{bmatrix}.$$

The diagonal matrices $\Lambda_r = \mathrm{diag}(r_1, \ldots, r_K)$ and $\Lambda_\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_K)$ in (5) contain the set of r and σ parameters for the ensemble of Lorenz oscillators, respectively. The vector $\mathbf{r} = (r_1, \ldots, r_K)$ contains the set of r parameters for each oscillator (the same parameters that correspond to the diagonal elements of Λ_r).

If we impose the restrictions $PB + C^T\Lambda_\sigma = 0$ and $\Lambda_\sigma D = D^T\Lambda_\sigma$, then $\dot{V}(\mathbf{v})$ can be written in the form

$$\dot{V}(\mathbf{v}) = -\mathbf{x}^T\Lambda_\sigma\Lambda_r\mathbf{x} - \mathbf{y}^T\Lambda_\sigma\mathbf{y} - \begin{bmatrix} \mathbf{z}-\mathbf{r} \\ \mathbf{l}-\mathbf{q} \end{bmatrix}^T M \begin{bmatrix} \mathbf{z}-\mathbf{r} \\ \mathbf{l}-\mathbf{q} \end{bmatrix} + c\,,$$

where the matrix M is given by

$$M = \begin{bmatrix} \Lambda_\sigma(\Lambda_b - D) & 0 \\ 0 & -\frac{1}{2}(PA + A^TP) \end{bmatrix}.$$

Also, the scalar c is given by

$$c = \mathbf{r}^T\Lambda_\sigma(\Lambda_b - D)\mathbf{r} - \mathbf{q}^T\frac{(PA + A^TP)}{2}\mathbf{q}\,,$$

and the vector $\mathbf{q}$ is given by

$$\mathbf{q} = -(PA + A^TP)^{-1}(PB - C^T\Lambda_\sigma)\mathbf{r}\,.$$

If M is positive definite and $c > 0$, then $\dot{V}(\mathbf{v}) = 0$ determines an ellipsoid in state space. Sufficient conditions for M to be positive definite and for $c > 0$ are given below.

- $PB + C^T\Lambda_\sigma = 0$.
- $\Lambda_\sigma D = D^T\Lambda_\sigma$.
- $PA + A^TP$ is negative definite.
- $\Lambda_\sigma(\Lambda_b - D)$ is positive definite.

The first and second conditions are simply the imposed restrictions. The third condition can be satisfied by choosing a stable A matrix such that $PA + A^TP$ is negative definite. The fourth condition provides a bound on D.

If these conditions are satisfied, then $\dot{V} = 0$ determines an ellipsoid of the form

$$\frac{\mathbf{x}^T\Lambda_\sigma\Lambda_r\mathbf{x}}{c} + \frac{\mathbf{y}^T\Lambda_\sigma\mathbf{y}}{c} + \frac{1}{c}\begin{bmatrix} \mathbf{z}-\mathbf{r} \\ \mathbf{l}-\mathbf{q} \end{bmatrix}^T M \begin{bmatrix} \mathbf{z}-\mathbf{r} \\ \mathbf{l}-\mathbf{q} \end{bmatrix} = 1\,. \quad (6)$$

Since $\dot{V} < 0$ for all $\mathbf{v}$ outside of the ellipsoid (6), any ellipsoid from the family (5) which contains (6) will suffice as a trapping region for the flow.

It is also important to determine the appropriate conditions which ensure that the transmitter equations are dissipative. The divergence of the transmitter's vector field is given by

$$\nabla \cdot \dot{\mathbf{v}} = \sum_{i=1}^{K}\left\{\frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \dot{y}_i}{\partial y_i} + \frac{\partial \dot{z}_i}{\partial z_i}\right\} + \sum_{i=1}^{N}\frac{\partial \dot{l}_i}{\partial l_i} = -(\mathrm{tr}(\Lambda_\sigma) + K + \mathrm{tr}(\Lambda_b - D) - \mathrm{tr}(A))\,.$$

The divergence is a negative constant if the condition

$$\mathrm{tr}(\Lambda_\sigma) + K + \mathrm{tr}(\Lambda_b - D) - \mathrm{tr}(A) > 0\,,$$

is satisfied. This condition alone ensures that the transmitter equations are dissipative with exponentially fast volume contraction. Below, we summarize the various self-synchronization and global stability conditions and suggest a straightforward synthesis procedure.

2.3. *A systematic synthesis procedure*

Sections 2.1 and 2.2 give sufficient conditions for the transmitter array to be dissipative and globally stable and for the receiver array to possess the global self-synchronization property. These conditions are summarized below.

Self-Synchronization
1. $RB + C^T = 0$, for some $N \times N$ positive definite matrix R.
2. $RA + A^T R$ is negative definite.
3. $\Lambda_b - \frac{1}{2}(D + D^T)$ is positive definite.

Global Stability
4. $PB + C^T \Lambda_\sigma = 0$, for some $N \times N$ positive definite matrix P.
5. $\Lambda_\sigma D$ is symmetric.
6. $PA + A^T P$ is negative definite.
7. $\Lambda_\sigma(\Lambda_b - D)$ is positive definite.

Dissipative
8. $\mathrm{tr}(\Lambda_\sigma) + K + \mathrm{tr}(\Lambda_b - D) - \mathrm{tr}(A) > 0$.

Although satisfying each of these conditions may seem to be a formidable task, the conditions can be significantly reduced by making two simplifying assumptions.

If we choose $P = \sigma R$ and $\Lambda_\sigma = \sigma I$, where I denotes the $K \times K$ identity matrix, then conditions 1 and 4 and conditions 2 and 6 are equivalent. Furthermore, condition 5 will then imply that D is symmetric, and thus, conditions 3 and 7 are equivalent. Also, conditions 2 and 6 imply that A is stable. In this case, condition 8 will be automatically satisfied. As a result of A being stable, there exists a positive definite solution R to the matrix Lyapunov equation

$$RA + A^T R + Q = 0\,, \quad Q > 0\,.$$

By choosing any stable A matrix and any symmetric positive definite Q matrix, conditions 2 and 6 can always be satisfied. In light of these simplifications, the following synthesis procedure is suggested.

Synthesis Procedure

1. Choose any stable A matrix and any $N \times N$ symmetric positive definite matrix Q.
2. Solve $RA + A^T R + Q = 0$ for the positive definite solution R.
3. Choose any $N \times K$ matrix B and set $C = -B^T R$.
4. Choose any $K \times K$ symmetric matrix D such that $\Lambda_b - D$ is positive definite.

The local stability of the equilibrium points should also be addressed. In Cuomo [1993b], a detailed linear stability analysis of a chaotic array consisting of two Lorenz oscillators and an N-dimensional linear system was performed. This analysis determined regions in (r_1, r_2) parameter space for which all of the equilibrium points are unstable. Choosing r_1 and r_2 within these regions of parameter space will ensure that the transmitter array exhibits nontrivial motion. In the next section, a numerical example demonstrates the nonlinear dynamical behavior of a typical low-order chaotic array.

3. Synthesis Example with Numerical Experiments

For the purpose of demonstration, consider the following 7-dimensional transmitter array.

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1)\,,\\
\dot{y}_1 &= r_1 x_1 - y_1 - x_1 z_1\,,\\
\dot{z}_1 &= x_1 y_1 - b_1 z_1 + o_1\,,\\
\dot{x}_2 &= \sigma(y_2 - x_2)\,,\\
\dot{y}_2 &= r_2 x_2 - y_2 - x_2 z_2\,,\\
\dot{z}_2 &= x_2 y_2 - b_2 z_2 + o_2\,,\\
\dot{l} &= -l + [-.36 \quad .97]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix},
\end{aligned}$$

$$\begin{bmatrix} o_1 \\ o_2 \end{bmatrix} = \begin{bmatrix} .36 \\ -.97 \end{bmatrix} l + \begin{bmatrix} .87 & -.10 \\ -.10 & .66 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

This array consists of two Lorenz oscillators and a one-dimensional linear system. Following the synthesis procedure outlined in Sec. 2.3, we chose $A = -1$, $Q = 2$, and randomly selected the elements of B and D from the normal distribution $N(0, 1)$. We then set $C = -B^T$ and verified that $\Lambda_b - D$ is positive definite. For the numerical experiments presented below, the Lorenz parameters $\sigma = 16$, $b_1 = 4$, and $b_2 = 4$ are fixed while the bifurcation parameters r_1 and r_2 are varied.

In Fig. 2, we show the stability diagram for this system. The stability diagram illustrates several regions in (r_1, r_2) parameter space where the chaotic array exhibits qualitatively different behavior. For example, the line segments (p, q) and (p', q') correspond to the boundaries where an abrupt change in the local stability of the array occurs. The

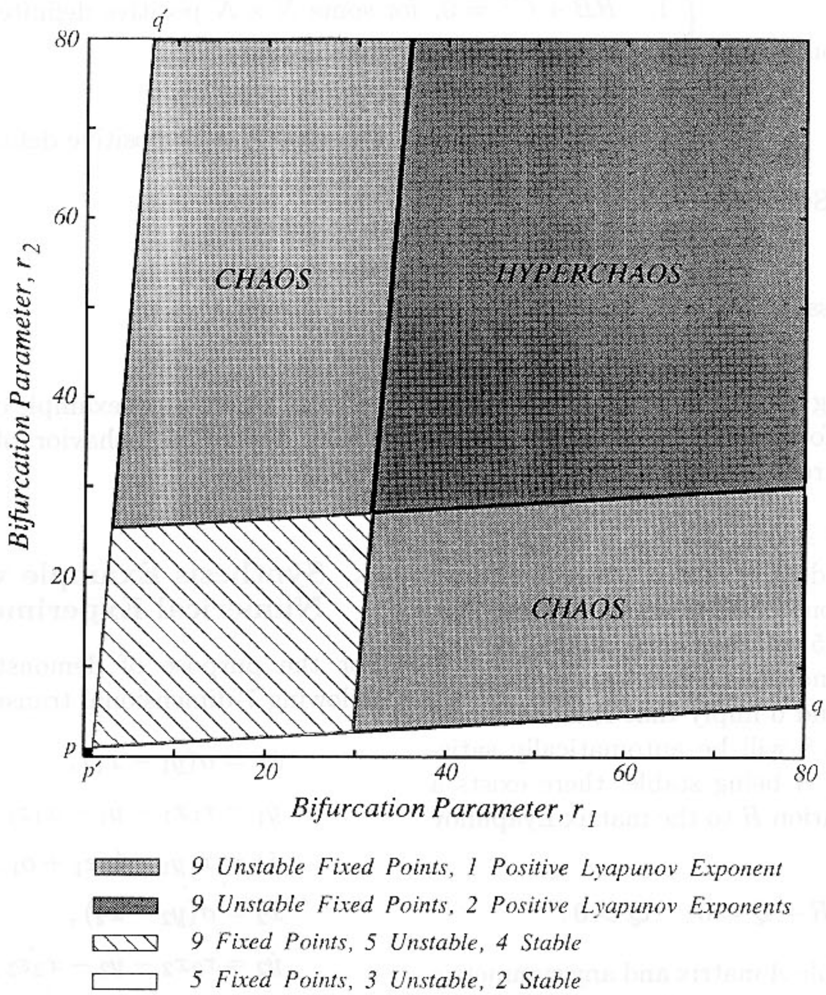

Fig. 2. Stability diagram for a 7-dimensional chaotic array.

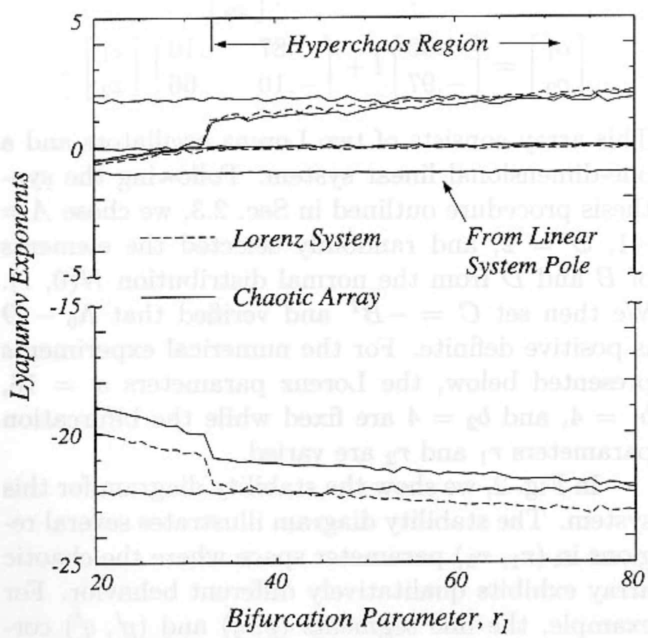

Fig. 3. Lyapunov exponents for a 7-dimensional chaotic array.

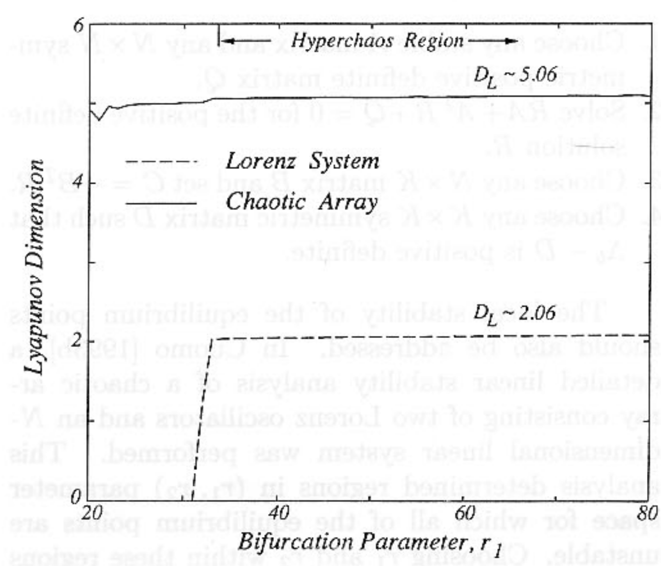

Fig. 4. Lyapunov dimension for a 7-dimensional chaotic array.

linear stability analysis performed in Cuomo [1993b] provides an exact mathematical representation of these boundaries. The shaded regions of the stability diagram indicate where chaotic motion occurs. The array exhibits a single positive Lyapunov exponent in the regions denoted "CHAOS," whereas two positive Lyapunov exponents exist in the regions denoted "HYPERCHAOS."

To visualize the dependence of the Lyapunov exponents on r_1, we show in Fig. 3 the Lyapunov spectrum as r_1 is varied over the range $20 < r_1 < 80$. The parameter r_2 in this experiment is held fixed at the value $r_2 = 60$. For $r_1 > 33$, two positive exponents exist (hyperchaos region); one corresponds to oscillator 1 and the other corresponds to oscillator 2. Several other important

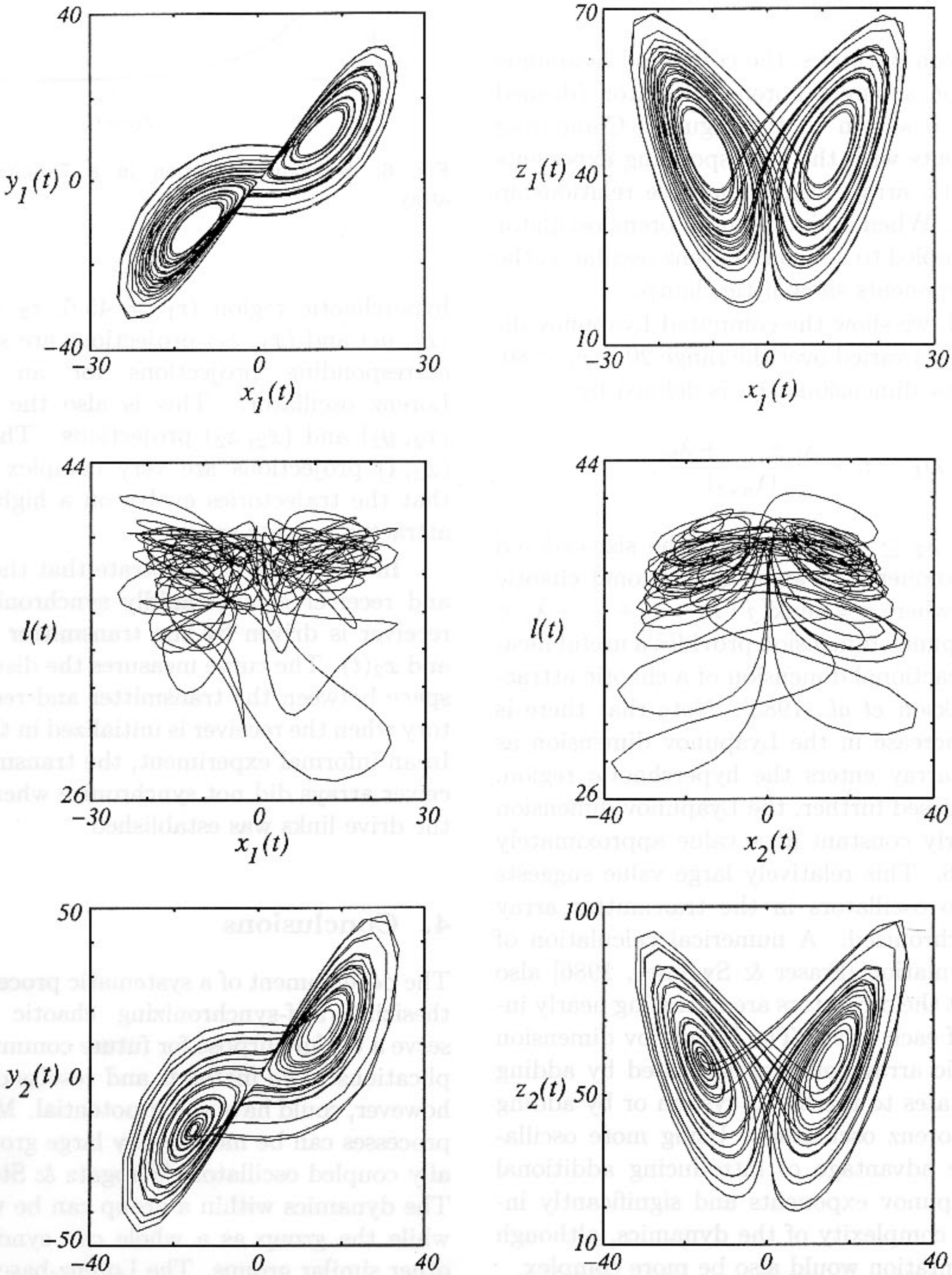

Fig. 5. Chaotic attractor projections for a 7-dimensional chaotic array.

features of the Lyapunov spectrum are listed below.

- An exponent equal to -1 is apparent. This exponent corresponds to the pole of the linear system.
- Two large negative exponents are apparent. These exponents are due to the highly dissipative nature of the chaotic array.
- Two zero exponents are apparent. These exponents correspond to motion tangent to the flow.

For comparison purposes, the computed Lyapunov exponents for a single Lorenz oscillator (dashed lines) are also shown in this figure. Comparing these exponents with the corresponding exponents for the chaotic array suggests a close relationship among them. When an individual Lorenz oscillator is linearly coupled to a second Lorenz oscillator, the Lyapunov exponents show little change.

In Fig. 4, we show the computed Lyapunov dimension as r_1 is varied over the range $20 < r_1 < 80$. The Lyapunov dimension, D_L, is defined by

$$D_L = n + \frac{\lambda_1 + \cdots + \lambda_n}{|\lambda_{n+1}|},$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ are the size-ordered Lyapunov exponents of an m-dimensional chaotic system, and where $n = \max\{j : \lambda_1 + \lambda_2 + \cdots + \lambda_j \geq 0\}$. The Lyapunov dimension provides a useful measure of the fractional dimension of a chaotic attractor [Frederickson *et al.*, 1983]. Note that there is an abrupt increase in the Lyapunov dimension as the chaotic array enters the hyperchaotic region. As r_1 is increased further, the Lyapunov dimension remains nearly constant at a value approximately equal to 5.06. This relatively large value suggests that the two oscillators in the transmitter array are not synchronized. A numerical calculation of mutual information [Fraser & Swinney, 1986] also suggests that the oscillators are operating nearly independent of each other. The Lyapunov dimension of the chaotic array could be increased by adding additional states to the linear system or by adding additional Lorenz oscillators. Using more oscillators has the advantage of introducing additional positive Lyapunov exponents and significantly increasing the complexity of the dynamics, although the implementation would also be more complex.

In Fig. 5, we show several projections of the chaotic array's attractor when operating in the

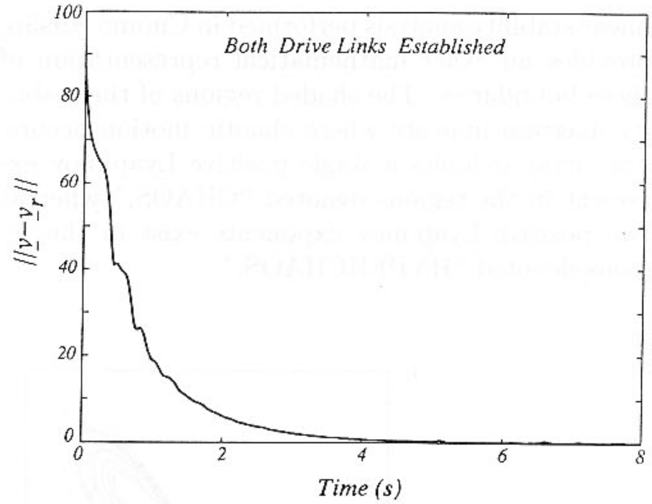

Fig. 6. Self-synchronization in a 7-dimensional chaotic array.

hyperchaotic region ($r_1 = 45.6$, $r_2 = 60$). The (x_1, y_1) and (x_1, z_1) projections are similar to the corresponding projections for an independent Lorenz oscillator. This is also the case for the (x_2, y_2) and (x_2, z_2) projections. The (x_1, l) and (x_2, l) projections are very complex and suggest that the trajectories evolve on a high-dimensional attractor in state space.

In Fig. 6, we demonstrate that the transmitter and receiver arrays rapidly synchronize when the receiver is driven by the transmitter signals $x_1(t)$ and $x_2(t)$. The curve measures the distance in state space between the transmitter and receiver trajectory when the receiver is initialized in the zero state. In an informal experiment, the transmitter and receiver arrays did not synchronize when only one of the drive links was established.

4. Conclusions

The development of a systematic procedure for synthesizing self-synchronizing chaotic arrays may serve a useful purpose for future communication applications. The methods and results of this paper, however, could have wider potential. Many physical processes can be modeled by large groups of mutually coupled oscillators [Strogatz & Stewart, 1993]. The dynamics within a group can be very complex while the group as a whole can synchronize with other similar groups. The Lorenz-based chaotic arrays investigated in this paper may lead to models useful for helping us to better understand these processes. Some conjectures and insights gained from this work are listed below.

- The individual Lorenz oscillators in a typical chaotic array operate nearly independent of each other. This conjecture is supported by the Lyapunov spectrum, Lyapunov dimension, and mutual information for a low-order chaotic array.
- It seems plausible that an array consisting of K Lorenz oscillators can exhibit K positive Lyapunov exponents. This conjecture is based on limited numerical experiments with low-order chaotic arrays.
- If communicating multiple drive signals is not an issue, then the recommended approach for synthesizing a complex transmitter and receiver array is to use as many Lorenz oscillators are possible and to couple them with a first-order linear system. This will produce an array with the most complex dynamics for a given state space dimension.
- If communicating multiple drive signals is a problem, then the dynamics of the transmitter and receiver arrays can be made more complex by using a larger linear system.
- Hardware implementations of chaotic arrays should be straightforward because of their modular structure.

The synthesis procedure provides the potential for designing high-dimensional self-synchronizing chaotic arrays which could be implemented in hardware and used in various private communication applications. We are currently exploring the applied aspects of these systems.

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References

Badola, P., Kumar, V. R. & Kulkarni, B. D. [1991] "Effects of coupling nonlinear systems with complex dynamics," *Phys. Lett.* **A155**, 365–372.

Carroll, T. L. & Pecora, L. M. [1991] "Synchronizing chaotic circuits," *IEEE Trans. Circuits and Systems* **38**, 453–456.

Cuomo, K. M. & Oppenheim, A. V. [1993] "Circuit implementation of synchronized chaos with applications to communications," *Phys. Rev. Lett.* **71**, 65–68.

Cuomo, K. M., Oppenheim, A. V. & Strogatz, S. H. [1993] "Synchronization of Lorenz-based chaotic circuits with applications to communications," *IEEE Trans. Circuits and Systems* **40**, 626–633.

Cuomo, K. M. [1993a] "Synthesizing self-synchronizing chaotic systems," *Int. J. Bifurcation and Chaos* **3**(5), 1327–1337.

Cuomo, K. M. [1993b] "Analysis and synthesis of self-synchronizing chaotic systems," Ph.D. Thesis, Massachusetts Institute of Technology.

Fraser, A. M. & Swinney, H. L. [1986] "Independent coordinates for strange attractors from mutual information," *Phys. Rev.* **A33**, 1134–1140.

Frederickson, P., Kaplan, J. L., Yorke, E. D. & Yorke, J. A. [1983] "The Liapunov dimension of strange attractors," *J. Diff. Eqns.* **49**, 185–207.

Halle, K. S., Wu, C. W., Itoh, M. & Chua, L. O. [1993] "Spread spectrum communication through modulation of chaos," *Int. J. Bifurcation and Chaos* **3**(2), 469–477.

Hansel, D. & Sompolinsky, H. [1992] "Synchronization and computation in a chaotic neural network," *Phys. Rev. Lett.* **68**, 718–721.

Klevecz, R. R., Bolen, J. & Duran, O. [1992] "Self-organization in biological tissues: Analysis of asynchronous and synchronous periodicity, turbulence and synchronous chaos emergence in coupled chaotic arrays," *Int. J. Bifurcation and Chaos* **2**(4), 941–953.

Kocarev, L., Halle, K. S., Eckert, K., Chua, L. O. & Parlitz, U. [1992] "Experimental demonstration of secure communications via chaotic synchronization," *Int. J. Bifurcation and Chaos* **2**(3), 709–713.

Kowalski, J. M., Albert, G. L. & Gross, G. W. [1990] "Asymptotically synchronous chaotic orbits in systems of excitable elements," *Phys. Rev.* **A42**, 6260–6263.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmos. Sci.* **20**, 130–141.

Oppenheim, A. V., Wornell, G. W., Isabelle, S. H. & Cuomo K. M. [1992] "Signal processing in the context of chaotic signals," in *Proc. 1992 IEEE ICASSP* **IV**, 117–120.

Parlitz, U., Chua, L. O., Kocarev, L., Halle, K. S. & Shang, A. [1992] "Transmission of digital signals by chaotic synchronization," *Int. J. Bifurcation and Chaos* **2**(4), 973–977.

Pecora, L. M. & Carroll, T. L. [1990] "Synchronization in chaotic systems," *Phys. Rev. Lett.* **64**, 821–823.

Pecora, L. M. & Carroll, T. L. [1991] "Driving systems with chaotic signals," *Phys. Rev.* **A44**, 2374–2383.

Rulkov, N. F., Volkovskii, A. R., Rodriguez-Lozano, A., Del Rio, E. & Velarde, M. G. [1992] "Mutual synchronization of chaotic self-oscillations with dissipative coupling," *Int. J. Bifurcation and Chaos* **2**(3), 669–676.

Strogatz, S. H. & Stewart, I. [1993] "Coupled oscillators and biological synchronization," *Scientific American* **269**, 102–109.

Waller, I. & Kapral, R. [1984] "Synchronization and chaos in coupled nonlinear oscillators," *Phys. Lett.* **A105**, 163–168.

Winful, H. G. & Rahman, L. [1990] "Synchronized chaos and spatiotemporal chaos in arrays of coupled lasers," *Phys. Rev. Lett.* **65**, 1575–1578.