Correspondence

A Separation Theorem for Periodic Sharing Information Patterns in Decentralized Control

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***Abstract*— Optimal decentralized control of a discrete-time stochastic system is considered under a periodic sharing information pattern. In this scenario, controllers share information with one-step delay every K time steps. The periodic sharing pattern is a generalization of the previously studied one-step delay sharing pattern, which is known to possess a nonclassical separation property. It is proven that the periodic sharing pattern has an analogous separation property.**

I. Introduction

Decentralized control refers to the regulation of complex systems by multiple controllers that make decisions based on different sensor information. The key element of a decentralized control problem is the information pattern, which characterizes the information available to the controllers at each time step. The information pattern not only determines the nature of optimal control laws but also greatly influences the difficulty of their design.

Various information patterns have been considered in the literature [1]–[6]. Perhaps the most widely studied information pattern has been the one-step delay sharing (OSDS) pattern, whereby each controller has instantaneous access to observations from its own measurement station as well as observations from all other stations after a one-step delay. It was shown that dynamic programming could be efficiently applied to the design of optimal control laws for the OSDS pattern after a nonclassical separation theorem was proven by Kurtaran [4] and Varaiya and Walrand [5]. Unfortunately, the OSDS pattern is often impractical because it requires that a large volume of data be shared with low delay. This low-delay requirement is relaxed under the n-step delay generalization of the OSDS pattern, but Varaiya and Walrand [5] showed that the n-step delay sharing pattern does *not* obey an analogous separation theorem.[1]

In this paper, we introduce periodic sharing information patterns, which constitute a different generalization of the OSDS pattern. Under a periodic sharing pattern, all controllers share observations with one-step delay every K time steps. While periodic sharing with period $K = 1$ corresponds to OSDS, the case in which no observations are shared is approached as $K \to \infty$. We prove that a separation property holds for periodic sharing patterns, thus admitting the possibility of using dynamic programming to efficiently design optimal control laws. Furthermore, for a particularly important class of systems, we show that this separation theorem implies that the volume of shared data is K times smaller under periodic sharing with period K than under either OSDS or n-step delay sharing.

In the next section, we formulate the optimal decentralized control problem under periodic sharing. In Section III, we state and prove a separation theorem that holds for periodic sharing patterns. In Section IV, we discuss the theorem and point out certain advantages, limitations, and potential applications of periodic sharing.

II. Problem Description

We shall consider a discrete-time stochastic system that is regulated by M decentralized controllers, each with an associated measurement station. The system state vector x_t and the mth measurement station's observations y_t^m evolve over T time steps according to the equations

$$x_{t+1} = f_t(x_t, u_t^1, \cdots, u_t^M, v_t) \qquad t = 0, 1, \cdots, T-1 \tag{1}$$

$$y_t^m = g_t^m(x_t, w_t^m), \qquad t = 0, 1, \cdots, T-1, \qquad m = 1, 2, \cdots, M \tag{2}$$

where f_t and g_t^m are given Borel-measurable functions, u_t^m represents the mth controller's input at time t, and the quantities $x_0, (v_0, w_0^1, \cdots, w_0^M), \cdots, (v_{T-1}, w_{T-1}^1, \cdots, w_{T-1}^M)$ are mutually independent random disturbance vectors whose distributions are known. At a particular time step t, the random vectors $v_t, w_t^1, \cdots, w_t^M$ may be statistically dependent. The vectors u_t^m, v_t, w_t^m, x_t, and y_t^m have given dimensions and take values in the given subsets U_t^m, V_t, W_t^m, X_t, and Y_t^m, respectively, all of which are Borel subsets of appropriately dimensioned Euclidean spaces.

Each controller produces, according to a predesigned control law, an input based on local observations from its own measurement station as well as observations that are shared every K time steps by all control stations. Since the sharing period has duration K, it is convenient to express any time $t \geq 0$ as $t = qK + r$, where q and r are integers such that $q \geq 0$ and $0 \leq r \leq K-1$. If the mth controller is governed at time t by the control law γ_t^m, then for $t = qK + r$ the control input is given by

$$u_{qK+r}^m = \gamma_{qK+r}^m(y_{qK}^m, y_{qK+1}^m, \cdots, y_{qK+r}^m, \delta_{qK+r}) \tag{3}$$

where $y_{qK}^m, \cdots, y_{qK+r}^m$ are local observations, and δ_{qK+r} is the shared information defined by (4), as shown at the bottom of the next page. Observe that since the sharing of information among controllers occurs only every K time steps, we have that $\delta_{qK} = \delta_{qK+1} = \cdots = \delta_{(q+1)K-1}$. Also note that $\delta_{(q+1)K}$ differs from δ_{qK} in that it contains all observations and controls generated during steps qK through $(q+1)K - 1$.

We denote the range of δ_{qK+r} by Δ_{qK+r} and define the set Γ_{qK+r}^m of admissible control laws for the mth controller at time $qK+r$ to be the set of Borel-measurable functions mapping $Y_{qK}^m \times \cdots \times Y_{qK+r}^m \times \Delta_{qK}$ to U_{qK+r}^m. The design objective is to choose the control laws

$$\gamma_t^m \in \Gamma_t^m, \qquad m = 1, \cdots, M, \qquad t = 0, \cdots, T-1 \tag{5}$$

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[1] Kurtaran [7] proved a modified "separation" theorem that can be exploited in certain cases to efficiently obtain optimal control laws [1]. This theorem shows that the estimator actually depends on the controller; hence, the two are not separated in the usual sense.

to minimize the total expected cost

$$\sum_{t=0}^{T-1} E[h_t(x_{t+1},\, u_t^1,\, \cdots,\, u_t^M)] \tag{6}$$

where h_t is a given Borel-measurable cost function. We shall assume that the expectation in (6) exists as an extended real number for all admissible control laws and also that (6) has a minimum over the admissible control laws. A control law achieving this minimum is called an *optimal control law*.[2] We also assume for convenience that T is an integer multiple of the sharing period K, that is, $T = NK$ for some integer N.

III. A Separation Theorem for Periodic Sharing Patterns

In this section, we prove that a separation property holds for the periodic sharing information pattern. More precisely, we prove that there is no loss in performance if the shared information $\delta_{qK+r} = \delta_{qK}$ is replaced by the function $F_{qK|\delta_{qK}}$ defined by $F_{qK|\delta_{qK}}(x_{qK}) = p(x_{qK}|\delta_{qK})$, where $p(x_{qK}|\delta_{qK})$ denotes the conditional probability density[3] of the state vector x_{qK} given δ_{qK}. A control law is said to be separated if it depends on δ_{qK} only through $F_{qK|\delta_{qK}}$. The set $\{F_{qK|\delta_{qK}} \,|\, \forall\, \delta_{qK} \in \Delta_{qK}\}$ is denoted by Φ_{qK}.

To prove the separation theorem, we consider the process of *actually finding* optimal control laws and show that the set of such laws has a nonempty intersection with the set of separated laws. We decompose the problem into N successive stages, each containing K time steps, and apply a modified version of the dynamic programming method outlined by Sandell and Athans [8]. Using the fact that the control laws at stage q do not affect the costs incurred at stages zero through $q-1$, we conclude that optimal control laws for the qth stage can be determined by minimizing a "cost-to-go" function. We show that this function depends only on the conditional density $F_{qK|\delta_{qK}}$, from which it follows that the control laws at stage q are separated. Finally, by induction we show that control laws for every stage are separated.

To ease the notational burden, we introduce some conventions before proceeding. First, we adopt the convention of using context to distinguish between values assumed by random variables and the random variables themselves. In addition, domains and ranges of functions are to be inferred from context but sometimes may be explicitly given for emphasis or clarity. To consolidate lists of related symbols, we define $y_t = (y_t^1,\, \cdots,\, y_t^M)$, $y_{s:t}^m = (y_s^m,\, \cdots,\, y_t^m)$, and $y_{s:t} = (y_s,\, \cdots,\, y_t)$. Moreover, we denote the range of y_t by $Y_t = \prod_{m=1}^M Y_t^m$, the range of $y_{s:t}^m$ by $Y_{s:t}^m = \prod_{j=s}^t Y_j^m$, and the range of $y_{s:t}$ by $Y_{s:t} = \prod_{j=s}^t Y_j$. Analogous notation will be used for the variables u_t, w_t, and γ_t, as well as their respective ranges. Finally, because densities and expectations generally depend on a choice of control laws $\gamma_{s:t}$, we denote the density and expectation induced by $\gamma_{s:t}$ as $p(\cdot;\, \gamma_{s:t})$ and $E[\cdot;\, \gamma_{s:t}]$, respectively.

For both the separation theorem and the lemma that precedes the theorem, it will be useful to define a control *sublaw* ψ_{qK+r}^m as a Borel-measurable function mapping $Y_{qK:qK+r}^m$ to U_{qK+r}^m. The set of all such sublaws is denoted Ψ_{qK+r}^m. Given a full control law $\gamma_{qK+r}^m \in \Gamma_{qK+r}^m$ and shared information $\delta_{qK} \in \Delta_{qK}$, we denote by $\gamma_{qK+r|\delta_{qK}}^m$ the sublaw satisfying

$$\gamma_{qK+r|\delta_{qK}}^m(y_{qK:qK+r}^m) = \gamma_{qK+r}^m(y_{qK:qK+r}^m,\, \delta_{qK}),\quad \forall\, y_{qK:qK+r}^m \in Y_{qK:qK+r}^m,\ \forall\, \delta_{qK} \in \Delta_{qK}. \tag{7}$$

The following lemma will be used to prove the separation theorem.

Lemma: For stage q of the optimization problem, define the K-step immediate cost function $C_q\colon \Gamma_{qK:(q+1)K-1} \times \Delta_{qK} \to \mathbb{R}$ by

$$\begin{aligned} &C_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}) \\ &\quad = E\left[\sum_{r=0}^{K-1} h_{qK+r}(x_{qK+r+1},\, u_{qK+r}) \,\middle|\, \delta_{qK};\, \gamma_{qK:(q+1)K-1}\right] \\ &\qquad \forall\, \delta_{qK} \in \Delta_{qK},\ \forall\, \gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1}. \end{aligned} \tag{8}$$

In addition, define the function $Q_q\colon \Gamma_{qK:(q+1)K-1} \times \Delta_{qK} \to \mathbb{R}$ by

$$\begin{aligned} &Q_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}) \\ &\quad = C_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}) \\ &\qquad + E[A_{q+1}(F_{(q+1)K|\delta_{(q+1)K}})|\delta_{qK};\, \gamma_{qK:(q+1)K-1}], \\ &\qquad \forall\, \delta_{qK} \in \Delta_{qK},\ \forall\, \gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1} \end{aligned} \tag{9}$$

where $A_{q+1}\colon \Phi_{(q+1)K} \to \mathbb{R}$ is a measurable function such that the conditional expectation above exists for all choices of control laws. Then, for all q such that $0 \le q \le N-1$, there exists a function $\tilde{Q}_q\colon \Psi_{qK:(q+1)K-1} \times \Phi_{qK} \to \mathbb{R}$ such that[4]

$$\begin{aligned} &Q_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}) = \tilde{Q}_q(\gamma_{qK:(q+1)K-1|\delta_{qK}},\, F_{qK|\delta_{qK}}), \\ &\qquad \forall\, \delta_{qK} \in \Delta_{qK},\ \forall\, \gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1}. \end{aligned} \tag{10}$$

Proof: See Appendix A. □

In the proof of the following separation theorem, the function A_{q+1} in (9) above will be a cost-to-go function.

Theorem: If there exists an optimal control law for the decentralized control problem under periodic sharing with period K, then there exist a particular optimal control law $\gamma_{0:NK-1}^*$ and corresponding functions $\{\phi_{qK+r}^m\colon Y_{qK:qK+r}^m \times \Phi_{qK} \to U_{qK+r}^m | 0 \le q \le N-1,\, 0 \le r \le K-1,\, 1 \le m \le M\}$ such that

$$\begin{aligned} &\gamma_{qK+r}^{*m}(y_{qK:qK+r}^m,\, \delta_{qK}) = \phi_{qK+r}^m(y_{qK:qK+r}^m,\, F_{qK|\delta_{qK}}), \\ &\qquad \forall\, \delta_{qK} \in \Delta_{qK},\ \forall\, y_{qK:qK+r}^m \in Y_{qK:qK+r}^m. \end{aligned} \tag{11}$$

Proof: Recall that the dynamic system operates for a total of $T = NK$ time steps. By modifying the dynamic programming equations put forth by Sandell and Athans [8], we arrive at the following set of recursively defined equations which will be used to characterize optimal control laws for the periodic sharing problem:

$$J_N^*(\delta_{NK}) = 0, \qquad \forall\, \delta_{NK} \in \Delta_{NK} \tag{12}$$

$$\begin{aligned} &J_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}) \\ &\quad = C_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}) \\ &\qquad + E[J_{q+1}^*(\delta_{(q+1)K})|\delta_{qK};\, \gamma_{qK:(q+1)K-1}], \\ &\qquad \forall\, \delta_{qK} \in \Delta_{qK},\ \forall\, \gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1} \end{aligned} \tag{13}$$

$$\begin{aligned} J_q^*(\delta_{qK}) = &\min_{\gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1}} \\ &J_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}), \qquad \forall\, \delta_{qK} \in \Delta_{qK}. \end{aligned} \tag{14}$$

[2] If a minimum does not exist, then one can only hope for ϵ-optimal control laws, but we will not address this case.

[3] Following [4], we work with probability densities throughout, but similar manipulations can be carried out using cumulative distribution functions.

[4] Saying a function $\tilde{a}$ exists such that $\tilde{a}(b(x)) = a(x)$ for all x is tantamount to saying that a depends on x only through $b(x)$.

$$\delta_{qK+r} = \begin{cases} (y_0^1,\, u_0^1,\, \cdots,\, y_0^M,\, u_0^M,\, \cdots,\, y_{qK-1}^1,\, u_{qK-1}^1,\, \cdots,\, y_{qK-1}^M,\, u_{qK-1}^M), & \text{if } q > 0 \\ 0, & \text{if } q = 0 \end{cases} \tag{4}$$

If a control law $\gamma^*_{0:NK-1} \in \Gamma_{0:NK-1}$ achieves the minimum in (14) for every $\delta_{qK} \in \Delta_{qK}$ and $0 \le q \le N-1$, then it minimizes the total expected cost in (6). The proof of this fact follows along the lines of the proof in [8] and is omitted here.

Let us now show that there exist functions $\tilde{J}^*_q$: $\Phi_{qK} \to \mathbb{R}$ and $\tilde{J}_q$: $\Psi_{qK:(q+1)K-1} \times \Phi_{qK} \to \mathbb{R}$ that satisfy

$$J^*_q(\delta_{qK}) = \tilde{J}^*_q(F_{qK|\delta_{qK}}), \qquad \forall\, \delta_{qK} \in \Delta_{qK} \tag{15}$$

$$\begin{aligned} &J_q(\gamma_{qK:(q+1)K-1},\, \delta_{qK}) \\ &\quad= \tilde{J}_q(\gamma_{qK:(q+1)K-1|\delta_{qK}},\, F_{qK|\delta_{qK}}) \\ &\qquad \forall\, \delta_{qK} \in \Delta_{qK},\ \forall\, \gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1}. \end{aligned} \tag{16}$$

The lemma and (13) together imply that if $\tilde{J}^*_{q+1}$ exists, then $\tilde{J}_q$ exists. Since it is clear that $\tilde{J}^*_N$ exists by (12), we have that $\tilde{J}_{N-1}$ also exists. To set up a proof by induction, suppose that $\tilde{J}_q$ exists. Then rewrite (14) as

$$\begin{aligned} J^*_q(\delta_{qK}) = &\min_{\psi_{qK:(q+1)K-1} \in \Psi_{qK:(q+1)K-1}} \\ &\tilde{J}_q(\psi_{qK:(q+1)K-1},\, F_{qK|\delta_{qK}}), \quad \forall\, \delta_{qK} \in \Delta_{qK}. \end{aligned} \tag{17}$$

Since the right-hand side of the above equation depends only on $F_{qK|\delta_{qK}}$, it follows that $\tilde{J}^*_q$, and therefore $\tilde{J}_{q-1}$, exist. By induction, it follows that $\tilde{J}_q$ and $\tilde{J}^*_q$ exist for all q such that $0 \le q \le N-1$.

Now, an optimal control law may be found as follows. Solve (17) for all $\delta \in \Delta_{qK}$ and $0 \le q \le N-1$. Let $\psi^{\delta}_{qK:(q+1)K-1} \in \Psi_{qK:(q+1)K-1}$ be the sublaw that achieves the minimum in (17) for δ at stage q, and construct control laws $\gamma^*_{qK:(q+1)K-1}$ according to

$$\begin{aligned} &\gamma^{*m}_{qK:(q+1)K-1}(y^m_{qK:(q+1)K-1},\, \delta) \\ &\quad= \psi^{\delta,m}_{qK:(q+1)K-1}(y^m_{qK:(q+1)K-1}), \qquad \forall\, \delta \in \Delta_{qK} \\ &\qquad \forall\, y^m_{qK:(q+1)K-1} \in Y^m_{qK:(q+1)K-1}. \end{aligned} \tag{18}$$

Then $\gamma^*_{0:NK-1}$ will be an optimal control law.[5]

To complete the proof, consider the sublaws minimizing (17). If $\psi^{\delta}_{qK:(q+1)K-1}$ achieves this minimum for $\delta \in \Delta_{qK}$, and there exists $\hat{\delta} \in \Delta_{qK}$ such that $F_{qK|\delta} = F_{qK|\hat{\delta}}$, then $\psi^{\delta}_{qK:(q+1)K-1}$ also achieves the minimum for $\hat{\delta}$. This proves that optimal control laws having the form given in (11) can be chosen. □

Remark 1: The dynamic programming equations (12)–(14) show that the control laws at a particular stage q must be found jointly, but that they may be found independently of the control laws for all other stages. If this were not the case, the complexity of searching for optimal control laws would increase enormously.

Remark 2: The period of sharing need not remain fixed in order for the separation property to hold. That is, the separation property will still hold if, for example, controllers share information after five steps, again two steps later, again eight steps later, and so on, as long as these sharing times are known in advance, before the control law is designed.

IV. Discussion and Conclusions

We have introduced a new class of information patterns in which controllers share observations every K time steps. For these new information patterns, known as periodic sharing patterns, we have shown that a nonclassical separation property holds. Since the separation property offers the possibility of efficient control law design via dynamic programming, attention is naturally drawn to the attributes of this new information pattern. We therefore devote this section to discussing the impact of the periodic sharing information pattern on the communication requirements of the system, the optimal cost of control, and the design complexity of optimal control laws, in comparison with the OSDS and n-step delay sharing patterns.

Periodic sharing patterns constitute a rich generalization of certain information patterns that have already received much study. For example, if we put $K = 1$, we have the OSDS pattern; on the other hand, as $K \to \infty$, we approach the case in which there is no information sharing. Freedom to choose K in the range $1 \le K < \infty$ affords great flexibility in the communication structures used in decentralized control systems. For example, imagine a team of people that makes decentralized decisions every hour. Meeting every hour to exchange information may be troublesome and wasteful. Instead, they may prefer to meet once a day (that is, every 24 h, so that $K = 24$) or once a week ($K = 7 \cdot 24$), at which point they share all the information currently available to them. Compare this with OSDS or n-step delay sharing, in which they must meet every hour. The parameter K can be chosen to minimize the total time-averaged cost of information sharing, which includes both the cost of establishing a communication link among controllers (for example, arranging a meeting) every K steps and the cost of each controller's data transmission once the link has been established. Note that in general, as K increases, a resulting increase in minimum cost [as measured by (6)] and increase in control law design complexity must be taken into account. Also note that, in general, the total volume of data shared under periodic sharing is the same as that under OSDS and n-step delay sharing. There is, however, an important class of systems for which the total volume of data shared under periodic sharing is K times *smaller* than the volume of data shared under OSDS or n-step delay sharing. This is the class of systems in which controllers receive noiseless observations. For such systems, it is assumed that the state vector x_t can be determined exactly from the complete set of observations $y^1_t, \cdots, y^M_t$. The separation theorem proven in Section III then implies that the controllers need only share the $2M$ quantities $(y_{qK-1},\, u_{qK-1})$, rather than all $2KM$ quantities $(y_{(q-1)K:qK-1},\, u_{(q-1)K:qK-1})$. That is, if only these $2M$ quantities are shared, the behavior of the optimally controlled system is the same as when all $2KM$ quantities are shared. It should be noted that in this case of noiseless observations, the periodic sharing pattern belongs to the class of patterns considered by Aicardi *et al.* [1]. However, in light of the separation theorem, periodic sharing patterns now warrant special consideration because their communication requirements (and design complexity) are lower than those implied by the solution techniques described in [1].

Next, we compare the minimum cost associated with a periodic sharing pattern to that associated with an OSDS or n-step delay pattern. The minimum cost associated with a periodic sharing pattern with period K is always less than or equal to the minimum cost associated with an n-step delay sharing pattern with $n \ge K$, but always greater than or equal to the minimum cost associated with the OSDS pattern. This follows once we recognize that under a periodic sharing pattern, each controller has access to shared observations that are at most K steps old, but at least one step old. Thus, periodic sharing with period K achieves better performance than K-step delay sharing.

Finally, let us compare the complexity of designing control laws under periodic sharing with the complexity of designing control laws under OSDS and n-step delay. In general, it is not known how to design control laws for arbitrary systems with any of these information patterns. For this reason, we focus on the case of finite state-space, finite-decision space systems with noiseless observations, for which we do know how to find control laws [1], [3]. In this case, designing an optimal control law for NK steps under a periodic sharing pattern with period K takes about N times the computation of designing a K-step optimal control law for a system with no

[5] As in [4] and [5], our treatment in this paper avoids measure-theoretic issues such as whether $\gamma^*_{0:NK-1}$ are Borel-measurable functions.

information sharing. Although the details are too involved to present here, it can also be shown that the design complexity under periodic sharing with period K is about the same as that under a K-step delay information pattern [1].

We conclude by mentioning an important application of the results of this correspondence. Using the separation property of periodic sharing patterns, we have derived upper bounds on the throughput of the multiple access broadcast channel, which can be modeled as a decentralized control system with no information sharing. Previously, researchers have exploited the separation property of the OSDS pattern to derive similar bounds [9]. However, because the periodic sharing pattern approaches the no-sharing pattern as $K \to \infty$, the bounds we have derived are tighter than those derived in [9]. Details will be reported in a future submission.

Appendix
Proof of Lemma

Before proving the main lemma, it is useful to introduce the following notation: define an expansion of a sublaw into a full control law as follows. For each $\psi^m_{qK+r} \in \Psi^m_{qK+r}$, define $G^m_{qK+r}(\psi^m_{qK+r}) \in \Gamma^m_{qK+r}$ to be the control law γ^m_{qK+r} satisfying

$$\gamma^m_{qK+r}(y^m_{qK:qK+r}, \delta_{qK}) = \psi^m_{qK+r}(y^m_{qK:qK+r}),$$
$$\forall y^m_{qK:qK+r} \in Y^m_{qK:qK+r}, \forall \delta_{qK} \in \Delta_{qK}. \tag{19}$$

Note that the range of G^m_{qK+r} is the set of control laws that vary only with local observations $y^m_{qK:qK+r}$ and remain constant with respect to δ_{qK}. We abbreviate G^m_{qK+r} by G, leaving both the superscript and subscript on G to be inferred from its argument. Also, we denote $(G(\psi^1_t), \cdots, G(\psi^M_t))$ by $G(\psi_t)$, and $(G(\psi_s), \cdots, G(\psi_t))$ by $G(\psi_{s:t})$.

To prove the main lemma, we will use three additional lemmas. We begin with a lemma that shows that the probability density for the random variables beyond the shared information δ_{qK} depends only on $F_{qK|\delta_{qK}}$.

Lemma 1: There exists a function $\tilde{\pi}$ such that

$$p(x_{j:j+K}, y_{j:j+K}|\delta_j; \gamma_{0:T-1})$$
$$= \tilde{\pi}(x_{j:j+K}, y_{j:j+K}, F_{j|\delta_j}, \gamma_{j:j+K-1|\delta_j}) \tag{20}$$

where $j = qK$, for $q = 0, \cdots, N-1$.

Proof: We use an inductive argument in which the main tool is the chain rule for probability density functions. If we put $j = qK$, then the last link of the chain can be written [4]

$$p(x_j, y_j|\delta_j; \gamma_{j:j+K-1}) = F_{j|\delta_j}(x_j) \cdot p(y_j|x_j). \tag{21}$$

Using the system equations (1) and (2), along with the fact that the random disturbance variables are independent at distinct times, and the fact that the control laws $G(\gamma_{j:j+K-1|\delta_j})$ return a constant control input over their last argument δ_j, we combine the last two links in the chain by

$$p(x_{j:j+1}, y_{j:j+1}|\delta_j; \gamma_{j:j+K-1})$$
$$= F_{j|\delta_j}(x_j) \cdot p(y_j|x_j) \cdot p(x_{j+1}|\delta_j, x_j, y_j; G(\gamma_{j|\delta_j}))$$
$$\cdot p(y_{j+1}|\delta_j, x_{j:j+1}, y_j) \tag{22}$$
$$= F_{j|\delta_j}(x_j) \cdot p(y_j|x_j) \cdot p(x_{j+1}|x_j, y_j; G(\gamma_{j|\delta_j}))$$
$$\cdot p(y_{j+1}|x_{j+1}). \tag{23}$$

Since (23) only depends on $F_{j|\delta_j}$, we conclude that a function $\tilde{\pi}_1$ exists such that

$$p(x_{j:j+1}, y_{j:j+1}|\delta_j; \gamma_{j:j+K-1})$$
$$= \tilde{\pi}_1(x_{j:j+1}, y_{j:j+1}, F_{j|\delta_j}, \gamma_{j|\delta_j}). \tag{24}$$

Let us now set up a proof by induction by supposing that there exists a function $\tilde{\pi}_s$ such that

$$p(x_{j:j+s}, y_{j:j+s}|\delta_j; \gamma_{j:j+K-1})$$
$$= \tilde{\pi}_s(x_{j:j+s}, y_{j:j+s}, F_{j|\delta_j}, \gamma_{j:j+s-1|\delta_j}) \tag{25}$$

for some s such that $1 \le s \le K-1$. Using the same argument used to derive (23), we write

$$p(x_{j+s+1}, y_{j+s+1}|\delta_j, x_{j:j+s}, y_{j:j+s}; \gamma_{j:j+K-1})$$
$$= p(x_{j+s+1}|\delta_j, x_{j:j+s}, y_{j:j+s}; G(\gamma_{j+s|\delta_j}))$$
$$\cdot p(y_{j+s+1} \mid \delta_j, x_{j:j+s+1}, y_{j:j+s}) \tag{26}$$
$$= p(x_{j+s+1}|x_{j+s}, y_{j+s}; G(\gamma_{j+s|\delta_j}))$$
$$\cdot p(y_{j+s+1}|x_{j+s+1}). \tag{27}$$

Since (27) depends on δ_j only through $\gamma_{j+s|\delta_j}$, we can use the chain rule to write

$$p(x_{j:j+s+1}, y_{j:j+s+1}|\delta_j; \gamma_{j:j+K-1})$$
$$= \tilde{\pi}_s(x_{j:j+s}, y_{j:j+s}, F_{j|\delta_j}, \gamma_{j:j+s-1|\delta_j}) \cdot (*) \tag{28}$$

where $(*)$ represents the right-hand side of (27). Then (28) implies that there exists a function $\tilde{\pi}_{s+1}$ such that

$$p(x_{j:j+s+1}, y_{j:j+s+1}|\delta_j; \gamma_{j:j+K-1})$$
$$= \tilde{\pi}_{s+1}(x_{j:j+s+1}, y_{j:j+s+1}, F_{j|\delta_j}, \gamma_{j:j+s|\delta_j}). \tag{29}$$

By induction, it follows that there exists a function $\tilde{\pi} = \tilde{\pi}_K$ such that

$$p(x_{j:j+K}, y_{j:j+K}|\delta_j; \gamma_{j:j+K-1})$$
$$= \tilde{\pi}_K(x_{j:j+K}, y_{j:j+K}, F_{j|\delta_j}, \gamma_{j:j+K-1|\delta_j}) \tag{30}$$

which is what we wished to show. □

Remark: Note that many of the densities in subsequent proofs can be derived from (20) via integration over the appropriate variables.

Next, we prove that the conditional density $F_{(q+1)K|\delta_{(q+1)K}}$ can be recursively obtained from $F_{qK|\delta_{qK}}$.

Lemma 2: There exists a mapping Υ_q: $\Phi_{qK} \times Y_{qK:(q+1)K-1} \times U_{qK:(q+1)K-1} \to \Phi_{(q+1)K}$ such that

$$F_{(q+1)K|\delta_{(q+1)K}}$$
$$= \Upsilon_q(F_{qK|\delta_{qK}}, y_{qK:(q+1)K-1}, u_{qK:(q+1)K-1}). \tag{31}$$

Proof: We argue that the mapping Υ_q exists by extending the arguments given in [4]. Let $j = qK$, and note that $F_{j+K|\delta_{j+K}}(x_{j+K}) = p(x_{j+K}|\delta_j, y_{j:j+K-1}, u_{j:j+K-1})$. Let $\gamma_{j:j+K-1}$ be control laws such that $\gamma^m_{j+s}(Y^m_{j:j+s} \times \Delta_j) = \{u^m_{j+s}\}$ for $m = 1, \cdots, M$ and $s = 0, \cdots, K-1$. That is, the control law γ^m_{j+s} always outputs u^m_{j+s} as the control input. We can then write

$$p(x_{j+K}|\delta_j, y_{j:j+K-1}, u_{j:j+K-1})$$
$$= p(x_{j+K}|\delta_j, y_{j:j+K-1}; \gamma_{j:j+K-1}). \tag{32}$$

Using the definition of conditional probability, we can write

$$p(x_{j+K}|\delta_j, y_{j:j+K-1}; \gamma_{j:j+K-1})$$
$$= \frac{p(x_{j+K}, y_{j:j+K-1}|\delta_j; \gamma_{j:j+K-1})}{p(y_{j:j+K-1}|\delta_j; \gamma_{j:j+K-1})}. \tag{33}$$

From Lemma 1, the right-hand side of (33) depends on δ_j only through $F_{j|\delta_j}$. Since this dependence holds for every $x_{j+K} \in X_{j+K}$, we conclude that $F_{j+K|\delta_{j+K}}$ can be expressed as in (31). □

Lemma 3: There exists a function $\tilde{C}_q$: $\Psi_{qK:(q+1)K-1} \times \Phi_{qK} \to \mathbb{R}$ such that

$$C_q(\gamma_{qK:(q+1)K-1}, \delta_{qK}) = \tilde{C}_q(\gamma_{qK:(q+1)K-1|\delta_{qK}}, F_{qK|\delta_{qK}}),$$
$$\forall \delta_{qK} \in \Delta_{qK}, \forall \gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1}. \tag{34}$$

Proof: We show using basic probability manipulations that the right-hand side of (8) depends only on $F_{qK|\delta_{qK}}$ and $\gamma_{qK:(q+1)K-1|\delta_{qK}}$. For notational convenience, we define a function ζ by

$$\zeta(x_{qK+1:(q+1)K},\, u_{qK:(q+1)K-1}) = \sum_{r=0}^{K-1} h_{qK+r}(x_{qK+r+1},\, u_{qK+r}). \tag{35}$$

In addition, for the control sublaws $\psi_{qK:qK+r} \in \Psi_{qK:qK+r}$, we will use the notation

$$\begin{aligned}\psi_{qK:qK+r}(y_{qK:qK+r}) = (&\psi^1_{qK}(y^1_{qK}), \cdots, \psi^M_{qK}(y^M_{qK}),\\ &\cdots, \psi^1_{qK+r}(y^1_{qK:qK+r})\\ &\cdots, \psi^M_{qK+r}(y^M_{qK:qK+r})).\end{aligned} \tag{36}$$

Using ζ, we write the expected K-step immediate cost explicitly as

$$\begin{aligned}E&\left[\sum_{r=0}^{K-1} h_{qK+r}(x_{qK+r+1},\, u_{qK+r})|\delta_{qK};\, \gamma_{qK:(q+1)K-1}\right]\\ &= \int \zeta(x_{qK+1:(q+1)K},\, \gamma_{qK:(q+1)K-1|\delta_{qK}}(y_{qK:(q+1)K-1}))\\ &\quad\cdot p(x_{qK+1:(q+1)K},\, y_{qK:(q+1)K-1}|\delta_{qK};\, \gamma_{qK:(q+1)K-1})\\ &\quad\cdot dx_{qK+1:(q+1)K}\cdot dy_{qK:(q+1)K-1} \qquad (37)\\ &= \int \zeta(x_{qK+1:(q+1)K},\, \gamma_{qK:(q+1)K-1|\delta_{qK}}(y_{qK:(q+1)K-1}))\\ &\quad\cdot \tilde{\eta}(x_{qK+1:(q+1)K},\, y_{qK:(q+1)K-1},\\ &\qquad F_{qK|\delta_{qK}},\, \gamma_{qK:(q+1)K-1|\delta_{qK}})\\ &\quad\cdot dx_{qK+1:(q+1)K}\cdot dy_{qK:(q+1)K-1} \qquad (38)\end{aligned}$$

where the existence of an appropriate $\tilde{\eta}$ in (38) follows from Lemma 1. Therefore, the right-hand side of (8) depends only on $F_{qK|\delta_{qK}}$ and $\gamma_{qK:(q+1)K-1|\delta_{qK}}$, and $\tilde{C}_q$ exists. □

We are now ready to prove the main lemma.

Proof of Main Lemma: We must show that Q_q depends only on $F_{qK|\delta_{qK}}$ and $\gamma_{qK:(q+1)K-1|\delta_{qK}}$. Since the first term of the right-hand side of (9) can be replaced by the right-hand side of (34), the proof reduces to showing that there exists a function $\tilde{\alpha}$ such that

$$\begin{aligned}E[A_{q+1}&(F_{(q+1)K|\delta_{(q+1)K}})|\delta_{qK};\, \gamma_{qK:(q+1)K-1}]\\ &= \tilde{\alpha}(F_{qK|\delta_{qK}},\, \gamma_{qK:(q+1)K-1|\delta_{qK}}),\\ &\quad \forall\, \delta_{qK} \in \Delta_{qK},\ \forall\, \gamma_{qK:(q+1)K-1} \in \Gamma_{qK:(q+1)K-1}.\end{aligned} \tag{39}$$

To show that (39) is true, we write

$$\begin{aligned}E[A_{q+1}&(F_{(q+1)K|\delta_{(q+1)K}})|\delta_{qK};\, \gamma_{qK:(q+1)K-1}]\\ &= \int A_{q+1}(\Upsilon_q(F_{qK|\delta_{qK}},\, y_{qK:(q+1)K-1},\\ &\qquad \gamma_{qK:(q+1)K-1|\delta_{qK}}(y_{qK:(q+1)K-1})))\\ &\qquad \cdot p(y_{qK:(q+1)K-1})|\delta_{qK};\, \gamma_{qK:(q+1)K-1})\\ &\qquad \cdot dy_{qK:(q+1)K-1}.\end{aligned} \tag{40}$$

Equation (40) follows from Lemma 2, and (40) with Lemma 1 implies that $\tilde{\alpha}$ satisfying (39) exists. □

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On d-Inversion in Interruptive Timed Discrete-Event Systems

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***Abstract*—The authors consider the problem of extracting event lifetimes from partial observations of an interruptive timed discrete-event system. The extraction of the lifetime of an occurring event is based on observations of all previous transitions and d subsequent transitions. We refer to this notion as d-inversion. We give necessary and sufficient structural conditions for an event to be d-invertible in a given system.**

***Index Terms*—Automata, discrete-event simulation, discrete-event systems, fault diagnosis, inverse problems, Markov processes, monitoring.**

I. INTRODUCTION

Considerable effort has been invested in the study of discrete event systems (DES's), due to their potential impact on the modeling, analysis, design, and control of a wide variety of complex systems, including discrete manufacturing systems, communication networks, traffic systems, and computer systems [6].

A key issue in DES's is the problem of gathering and analyzing observed data from a given system as it is evolving, often referred to as *on-line monitoring* (see [19] for an overview of the subject in the setting of distributed software systems). Observation data is used in a wide variety of contexts, including supervisory control [17], [9], [4], fault testing and diagnosis [18], [8], [1], performance analysis and

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