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# Information-theoretic analysis of multiple-antenna transmission diversity for fading channels

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*Abstract* — **Several simple schemes suggested in the literature for transmitter antenna diversity are put into a common framework and compared to a theoretically optimal system. The comparison is based on mutual information as a function of antenna element gains. We show that the suboptimal schemes are uniformly worse than the optimal scheme, though the gap is small when the spectral efficiency in bits per symbol is low.**

## I. Introduction

It is well-known [1] that multiple antennas can improve the performance of a communication system in a fading environment. These multiple antennas may be employed either at the transmitter or at the receiver. In a mobile radio system, it is most cost effective to employ multiple antennas at the base station and single or double antennas on the mobile units. Thus, in transmitting from the mobile to the base station, diversity is achieved through multiple receive antennas and in transmitting from the base station to the mobiles, diversity is achieved through multiple transmit antennas. In this paper we focus on transmitter diversity. A signal-processing approach to the same problem is considered in [2].

Transmitter diversity is generally viewed as more difficult to exploit than receiver diversity, in part because the transmitter is assumed to know less about the channel than the receiver, and in part because the transmitter is permitted to generate a different signal at each antenna. Unlike the receiver diversity case, where independently faded copies of a single transmitted signal may be combined optimally to achieve a performance gain, for transmitter diversity the many transmitted signals are already combined when they reach the receiver. How, then, should the transmitted signals be selected to either achieve capacity, or, more practically, to simplify the receiver while maintaining performance near capacity?

## II. Channel Model

We model the $M$-antenna transmitter diversity channel as shown in Figure 1. The complex baseband received signal $y_k = \sum_{i=1}^{M} \alpha_i x_{i,k} + v_k$ at time $k$ is the superposition of the $M$ transmitted symbols $x_{1,k}, \ldots, x_{M,k}$, each scaled and phase-shifted by a complex fading coefficient $\alpha_i$ which represents the aggregate effect of the channel encountered by antenna $i$. The channel is frequency nonselective, i.e., the delay spread of the channel is small compared to the symbol duration. The additive noise $v_k$ is assumed to be white circular Gaussian with variance (for each real and imaginary component) $N_0/2$, and the average transmitted energy is limited to $\sum_i E|x_{i,k}|^2 \leq \mathcal{E}_s$ per symbol. As $M$ increases the power must be distributed among the antenna elements; this allows a fair comparison of single and multiple antenna systems.

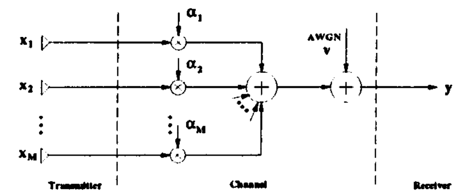


Fig. 1: Fading channel with $M$ transmit antennas.

The channel is assumed to be slowly varying, so that the fading coefficients $\{\alpha_i\}$ are effectively constant over the signaling interval of interest. The transmitter is assumed to have no knowledge of the fading coefficients, while the receiver is assumed to have perfect knowledge. We expect imprecise channel measurement to cause a smooth degradation in performance that is largely separable from other effects, though we have not established this result rigorously.

Following [3], our measure of performance is the mutual information between input and output over a long block, which corresponds in an approximate sense to the maximum achievable rate of reliable communication.

## III. Diversity Methods

We analyze five schemes for exploiting multiple transmitter antennas: "unconstrained" signaling, time division, frequency division, the time-shift technique proposed by Winters [4], and the frequency-shift technique proposed by Hiroike [5]. We will focus on the two-antenna case; the generalization to $M > 2$ is straightforward.

By unconstrained signaling we mean that the system is evaluated as a vector-input scalar-output Gaussian channel. The other four schemes, shown in Figure 2, use linear processing to convert the vector-input channel into a scalar-input channel.

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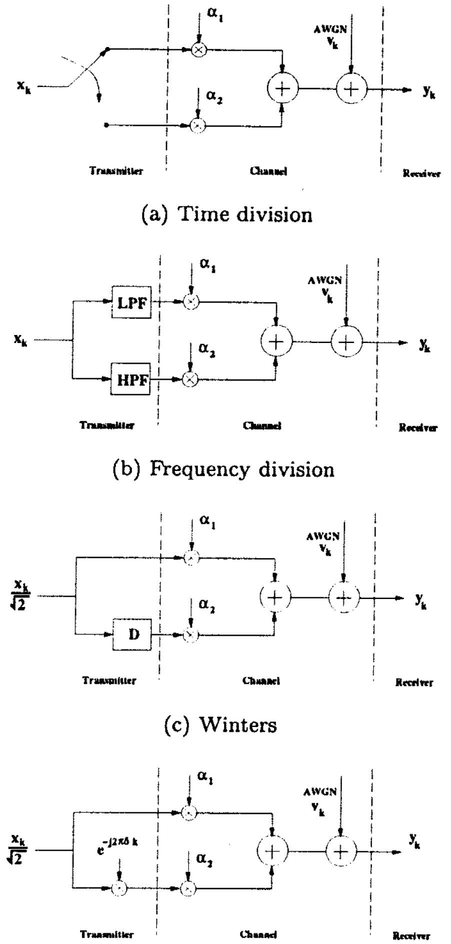
Let $x_k$ be the scalar input sequence. For time-division, antenna 1 is used for even $k$ and antenna 2 is used for odd $k$:

$$x_{1,k} = \begin{cases} x_k & \text{if } k \text{ even,} \\ 0 & \text{if } k \text{ odd,} \end{cases} \qquad x_{2,k} = \begin{cases} 0 & \text{if } k \text{ even,} \\ x_k & \text{if } k \text{ odd.} \end{cases}$$




(a) Time division

(b) Frequency division

(c) Winters

(d) Hiroike

Figure 2: Four transmitter diversity schemes.

For frequency division, antenna 1 transmits a lowpass filtered version of $x_k$ while antenna 2 transmits the complementary highpass filtered version of $x_k$. Any linear method that generates orthogonal signals $x_{1,k}$ and $x_{2,k}$ from $x_k$ (such that the signals remain orthogonal after passing through the channel) will have the same characteristic behavior as time and frequency-division.

For the methods of Winters and Hiroike, the first antenna carries the signal unaltered: $x_{1,k} = x_k$. Winters transmits a delayed copy of $x_k$ on the second antenna, $x_{2,k} = x_{k-1}$. Hiroike transmits a frequency-shifted copy of $x_k$, $x_{2,k} = e^{j2\pi\delta k}x_k$, for some $\delta > 0$. As with time and frequency division, the methods of Winters and Hiroike are time/frequency duals.

Frequency division and Winters' method convert antenna diversity into frequency diversity; the memoryless vector-input channel becomes a scalar-input channel with intersymbol interference. Time division and Hiroike's method convert antenna diversity into time diversity; the time invariant vector-input channel becomes a periodically time varying scalar-input channel.

## IV. Mutual information calculations

In this section we compute the mutual information achieved by the five diversity schemes described in Section III. We assume that the input codebooks are derived from i.i.d. complex circular Gaussian random processes. The use of Gaussian codebooks follows from the assumption that the receiver knows the channel. I.i.d. codebooks can be justified to some extent in a game theoretic sense; they are the saddle point solution to the max-min problem in which nature chooses the worst possible channel (for a fixed channel gain $|\alpha_1|^2 + |\alpha_2|^2$) and the transmitter chooses the best signaling scheme.

Time division and Hiroike's method, which create time varying channels, are analyzed in the time domain. Frequency division and Winters' method, which create intersymbol interference channels, are analyzed in the frequency domain. In all cases, beam forming and waterfilling methods cannot be applied because the transmitter is assumed to have no knowledge of the channel parameters.

### A. *"Unconstrained" signaling*

The unconstrained multiple transmit antenna system is a memoryless vector-input scalar-output power-limited Gaussian channel. With two transmit antennas, the complex baseband received signal is

$$Y = \alpha_1 X_1 + \alpha_2 X_2 + V, \tag{1}$$

where $X_1$ and $X_2$ are zero-mean complex Gaussian random variables with variance $\mathcal{E}_s/2$ each and $V$ is complex Gaussian noise with variance $N_0$. The output $Y$ is zero-mean Gaussian with variance $(|\alpha_1|^2 + |\alpha_2|^2)\frac{\mathcal{E}_s}{2} + N_0$. The mutual information for this channel is

$$I_{\text{DIV}} = h(Y) - h(Y|X_1, X_2) \tag{2}$$

$$= \log\left(1 + \frac{(|\alpha_1|^2 + |\alpha_2|^2)\mathcal{E}_s}{2N_0}\right). \tag{3}$$

The mutual information depends only on the antenna gain $|\alpha_1|^2 + |\alpha_2|^2$. Interestingly, the same formula applies to ideal beamforming, but without the factor of 2 in the denominator.

### B. *Time-division and frequency-division*

A simple way to exploit transmit antenna diversity is to transmit signals orthogonal in time or frequency on each antenna. In the absence of intersymbol interference or Doppler spread, the signals will remain orthogonal at the receiver. The multiple antenna channel can then be analyzed as a set of independent parallel channels. As in the case of the multiple access Gaussian channel, the mutual informations achieved by time division and frequency division are equal.

The time-division approach is shown in Figure 2(a). Odd-time inputs are transmitted using antenna 1 and even-time inputs are transmitted on antenna 2. The input $X_k$ is i.i.d. Gaussian with energy $E|X_k|^2 = \mathcal{E}_s$ per symbol. The output of this channel is

$$Y_k = \begin{cases} \alpha_1 X_k + V_k & k \text{ odd}, \\ \alpha_2 X_k + V_k & k \text{ even}, \end{cases} \tag{4}$$

and the mutual information between input and output is

$$I_{\mathrm{TD}} = \frac{1}{2}\log\left(1 + \frac{|\alpha_1|^2\mathcal{E}_s}{N_0}\right) + \frac{1}{2}\log\left(1 + \frac{|\alpha_2|^2\mathcal{E}_s}{N_0}\right). \tag{5}$$

The frequency-division approach is shown in Figure 2(b). The transmitter generates an i.i.d. Gaussian sequence $X_k$ that meets the power constraint $E|X_k|^2 = \mathcal{E}_s$. The sequence is lowpass filtered to bandwidth $\pi/2$ and transmitted over antenna 1; the sequence is highpass filtered to frequencies above $\pi/2$ and transmitted over antenna 2. The Fourier transform $H(\omega)$ of the unit-sample response of the resulting scalar-input channel is

$$H(\omega) = \begin{cases} \alpha_1 & |\omega| < \pi/2, \\ \alpha_2 & \pi/2 \le |\omega| < \pi. \end{cases} \tag{6}$$

The mutual information between input and output is

$$I_{\mathrm{FD}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(1 + \frac{\mathcal{E}_s|H(\omega)|^2}{N_0}\right) \tag{7}$$

$$= \frac{1}{2}\log\left(1 + \frac{|\alpha_1|^2\mathcal{E}_s}{N_0}\right) + \frac{1}{2}\log\left(1 + \frac{|\alpha_2|^2\mathcal{E}_s}{N_0}\right). \tag{8}$$

As expected, time division and frequency division signaling yield the same mutual information.

Application of Jensen's inequality to (8) shows that the mutual information achieved by time or frequency division is always less than or equal to that achieved by the unconstrained channel (3), with equality if and only if the two antenna gains $|\alpha_1|$ and $|\alpha_2|$ are equal.

### *C. Time-delayed transmit signals*

In Winters' [4] transmit diversity scheme, delayed versions of a common input signal are sent over the multiple transmit antennas. This simple linear processing converts the multiple antenna system to a scalar intersymbol interference channel.

The channel determined by Winters' scheme is shown in Figure 2(c). The output of the discrete-time channel is

$$Y_k = \alpha_1 X_k + \alpha_2 X_{k-1} + V_k, \tag{9}$$

where $X_k$ is i.i.d. complex Gaussian with variance $E|X_k|^2 = \mathcal{E}_s/2$. The Fourier transform of the unit-sample response of this channel is

$$H(\omega) = \alpha_1 + e^{j\omega}\alpha_2, \quad |\omega| \le \pi. \tag{10}$$

The frequency response of the channel varies from perfect coherent combining to perfect destructive interference between the two antenna elements. The mutual information is

$$I_{\mathrm{WIN}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(1 + \frac{(|\alpha_1 + e^{j\omega}\alpha_2|^2\mathcal{E}_s}{2N_0}\right)d\omega. \tag{11}$$

Again, by Jensen's inequality, the mutual information (11) achieved by Winters method is less or equal to that achieved by unconstrained signaling, with equality if and only if $\alpha_1 = 0$ or $\alpha_2 = 0$.

### *D. Phase modulated transmit signals*

Hiroike et al. [5] use multiple transmit antennas to send phase modulated versions of the same input signal. They assume a continuous-time system and modulate the signal by a sweeping function $e^{j2\pi\delta_i t}$ for the $i^{\mathrm{th}}$ antenna. In the frequency domain, this corresponds to sending frequency shifted versions of the transmitted signal. Hiroike's transmission scheme is the frequency domain analog of Winters' method analyzed in the previous section. Thus we expect the mutual information to be similar.

We analyze an equivalent discrete-time model of Hiroike's system, shown in Figure 2(d). To derive the mutual information of this periodic time-varying channel, we first compute the mutual information of an $N$-block channel and then take the limit as $N$ becomes infinite. For the two transmit antenna system, the channel output is

$$Y_k = \alpha_1 X_k + \alpha_2 e^{j2\pi k/N} X_k + V_k, \tag{12}$$

where the frequency shift parameter $\delta = 1/N$ has been chosen so that the phase of the second antenna rolls through one full cycle over the block of $N$ symbols. This channel is equivalent to $N$ parallel AWGN channels, one for each time $k$. The inputs $X_k$ are assumed to be complex Gaussian random variables with mean zero and variance $E|X_k^2| = \mathcal{E}_s/2$. The mutual information of the $N$-block channel is

$$I_N = \frac{1}{N}\sum_{k=0}^{N-1}\log\left(1 + \frac{|\alpha_1 + e^{j2\pi k/N}\alpha_2|^2\mathcal{E}_s}{2N_0}\right). \tag{13}$$

Taking the limit as $N \to \infty$ yields

$$I_{\text{HIR}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(1 + \frac{|\alpha_1 + e^{j\omega}\alpha_2|^2 \mathcal{E}_s}{2N_0}\right) d\omega, \quad (14)$$

which is the same as the mutual information achieved by Winters' method.

## V. Discussion

A useful way to interpret the mutual information formulas derived above is to plot lines of constant mutual information on the $|\alpha_1|$,$|\alpha_2|$ quarter plane. These curves delimit outage regions, in the sense that one can devise a codebook at any rate $R$ so that reliable communication will occur if the mutual information exceeds $R$. The curves for $I = 1$ are shown in Figure 3: the solid innermost quarter circle is unconstrained diversity $I_{\text{DIV}}$; the dashed curve tangent to the quarter circle at $|\alpha_1| = |\alpha_2|$ is time division $I_{\text{TD}}$ and frequency division $I_{\text{FD}}$; the dotted curve tangent to the quarter circle at $|\alpha_1| = 0$ and $|\alpha_2| = 0$ is Winters' method $I_{\text{WIN}}$ and Hiroike's method $I_{\text{HIR}}$.

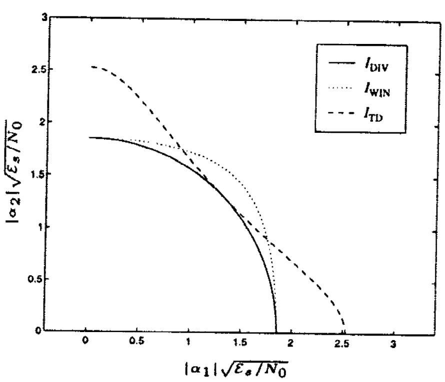



Fig. 3: Values of $|\alpha_1|\sqrt{\mathcal{E}_s/N_0}$ and $|\alpha_2|\sqrt{\mathcal{E}_s/N_0}$ that correspond to a mutual information of 1 bit/complex symbol.

The outage region for unconstrained signaling is uniformly smaller than the outage regions for any of the four suboptimal methods. The tradeoff between the suboptimal diversity methods is less clear. For example, the suboptimal diversity method that minimizes outage probability will depend on the stochastic model for $|\alpha_1|$ and $|\alpha_2|$. Though not indicated on the figure, the gap between the optimal and suboptimal approaches increases with $I$.

Time division and Hiroike's method can be implemented in continuous time at passband. This is a simple way to upgrade a system designed for a single antenna to use transmitter diversity, though of course the full performance gain will accrue only if the transmitter and receiver processing is redesigned for the (artificially created) time-varying channel. Coding methods used for broadcast HDTV or digital audio should be applicable here, with the simplification that the time-variation of the channel follows a pattern known to the receiver.

If the goal is to minimize outage probability then the restriction to four suboptimal diversity methods seems artificial. A more natural assumption, consistent with the goal of minimizing complexity, would be to consider the class of all linear periodic time-varying filters at the transmitter. Some work along these lines is presented in [2].

To achieve the full benefit of transmit diversity over the vector Gaussian channel requires more sophisticated coding and decoding. One approach is to insert a pseudo-randomly varying unitary matrix before the antenna array. The transmitter can then split itself into two virtual users, time-synchronized but otherwise noncooperative, and use multiple access coding. If the receiver uses successive decoding (stripping), the complexity should be within a reasonable factor of a single-antenna system. The channel identification and power-control problems normally associated with stripping are not as severe in this case, because the multiple virtual users arise from a single user.

An open problem in this area is to develop a framework for the comparative assessment of transmitter diversity, receiver diversity, and the value of partial channel knowledge at the receiver and transmitter. For example, it can be shown that a transmitter array using ideal beamforming has the same performance as a receiver array using ideal maximal ratio combining. In the absence of accurate channel knowledge, a receiver array still achieves some performance gain if noncoherent signaling is used; the transmitter array still achieves some performance gain if a diversity method is used.

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