

# THEORY AND IMPLEMENTATION OF THE DISCRETE HILBERT TRANSFORM

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The Hilbert transform has traditionally played an important part in the theory and practice of signal processing operations in continuous system theory because of its relevance to such problems as envelope detection and demodulation, as well as its use in relating the real and imaginary components, and the magnitude and phase components of spectra. The Hilbert transform plays a similar role in digital signal processing. In this paper, the Hilbert transform relations, as they apply to sequences and their  $z$ -transforms, and also as they apply to sequences and their discrete Fourier transforms, will be discussed. These relations are identical only in the limit as the number of data samples taken in the discrete Fourier transforms becomes infinite.

The implementation of the Hilbert transform operation as applied to sequences usually takes the form of digital linear networks with constant coefficients, either recursive or nonrecursive, which approximate an all-pass network with  $90^\circ$  phase shift, or two-output digital networks which have a  $90^\circ$  phase difference over a wide range of frequencies. Means of implementing such phase shifting and phase splitting networks are presented.

## 1. INTRODUCTION

Hilbert transforms have played a useful role in signal and network theory and have also been of practical importance in various signal processing systems. Analytic signals, bandpass sampling, minimum phase networks and much of spectral analysis theory is based on Hilbert transform relations. Systems for performing Hilbert transform operations have proved useful in diverse fields such as radar moving target indicators, analytic signal rooting [1], measurement of the voice fundamental frequency [2, 3], envelope detection, and generation of the phase of a spectrum given its amplitude [4, 5, 6].

The present paper is a survey of Hilbert transform relations in digital systems, and of the design of linear digital systems for performing the Hilbert transform of an input signal. These subjects have been treated in the published literature for continuous signals and systems [7]. In this paper we present a treatment of the subject for digital systems. We will first present various Hilbert transform relationships followed by several design techniques for Hilbert transformers and a few examples and applications.

## 2. CONVOLUTION THEOREMS

In this section some notation is introduced and some well-known convolution theorems are quoted; the interested reader can find proofs of these and other

theorems of  $z$ -transform theory in various books [8, 9]. Let  $x(n)$  be a stable infinite sequence and define the  $z$ -transform of  $x(n)$  as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} .$$

Given two such sequences  $x(n)$  and  $h(n)$  and their corresponding  $z$ -transforms  $X(z)$  and  $H(z)$ , then, if  $Y(z) = X(z)H(z)$ , we have the convolution theorem

$$y(n) = \sum_{m=-\infty}^{m=\infty} x(n-m) h(m) = \sum_{m=-\infty}^{m=\infty} x(m) h(n-m) . \quad (1)$$

Similarly, if  $y(n) = x(n) h(n)$ , we have the complex convolution theorem

$$Y(z) = \frac{1}{2\pi j} \oint X(v) H(z/v) v^{-1} dv = \frac{1}{2\pi j} \oint X(z/v) H(v) v^{-1} dv , \quad (2)$$

where  $v$  is the complex variable of integration and the integration path chosen is the unit circle, taken counterclockwise.

The spectrum of a signal is defined as the value of its  $z$ -transform on the unit circle in the  $z$ -plane. Thus, the spectrum of  $x(n)$  can be written as  $X(e^{j\theta})$  where  $\theta$  is the angle of the vector from the origin to a point on the unit circle.

If  $x(n)$  is a sequence of finite length  $N$ , then it can be represented by its discrete Fourier transform (DFT). Denoting the DFT values by  $X(k)$ , we have

$$X(k) = \sum_{n=0}^{N-1} x(n) W^{-nk} ,$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W^{nk} , \quad (3)$$

with

$$W = e^{j2\pi/N} .$$

The convolution theorems for these finite sequences specify that if  $Y(k) = H(k) X(k)$ , then

$$y(n) = \sum_{m=0}^{N-1} x(m) h(((n-m))) = \sum_{m=0}^{N-1} x(((n-m))) h(m) , \quad (4)$$

and if  $y(n) = x(n) h(n)$ , then

$$Y(k) = \frac{1}{N} \sum_{\ell=0}^{N-1} X(\ell) H(((k-\ell))) = \frac{1}{N} \sum_{\ell=0}^{N-1} X(((k-\ell))) H(\ell) , \quad (5)$$

where the double parenthesis around the expressions  $k-\ell$  and  $n-m$  refer to these expressions modulo  $N$ ; i.e.,  $((x))$  is the unique integer  $x+kN$ , satisfying  $0 \leq x+kN \leq N-1$ .

Finally we define an infinite sequence  $x(n)$  as “causal” if  $x(n) = 0$  for  $n < 0$ . A finite duration sequence of length  $N$  is “causal” if  $x(n)$  is zero in the latter half of the period  $0, 1, \dots, N - 1$ , i.e., for  $n > \frac{N}{2}$ .

### 3. HILBERT TRANSFORM RELATIONS FOR CAUSAL SIGNALS

The  $z$ -transform  $X(z)$  of the impulse response  $x(n)$  of a linear stable causal digital system is analytic outside the unit circle. Under these conditions, interesting relations between components of the complex function  $X(z)$  can be derived, these relations being a consequence of the Cauchy integral theorem [10]. For example,  $X(z)$  can be explicitly found outside the unit circle given either the real or imaginary components of  $X(z)$  on the unit circle. These relations also hold on the unit circle; if we write

$$X(e^{j\theta}) = R(e^{j\theta}) + jI(e^{j\theta}) \quad , \quad (6)$$

where  $R(e^{j\theta})$  and  $I(e^{j\theta})$  are the real and imaginary parts respectively of  $X(e^{j\theta})$ , then  $X(e^{j\theta})$  can be explicitly found in terms of  $R(e^{j\theta})$  or in terms of  $I(e^{j\theta})$  and therefore  $R(e^{j\theta})$  and  $I(e^{j\theta})$  can be expressed as functions of each other. These various integral relationships will be referred to as Hilbert transform relations between components of  $X(z)$ .

First, we will derive an expression for  $X(z)$  outside (not on) the unit circle given  $R(e^{j\theta})$  (on the unit circle), beginning with the physically appealing concept of causality. A causal sequence can always be reconstructed from its even part, defined as

$$x_e(n) = \frac{1}{2} [x(n) + x(-n)] \quad . \quad (7)$$

Since  $x(n)$  is causal, it can be written

$$\left. \begin{aligned} x(n) &= 2 x_e(n) s(n) \\ \text{where} \quad s(n) &= 1 \quad \text{for} \quad n > 0 \\ &= 0 \quad \text{for} \quad n < 0 \\ &= \frac{1}{2} \quad \text{for} \quad n = 0 \end{aligned} \right\} \quad . \quad (8)$$

Now, consider  $X(z)$  outside the unit circle, that is, for  $z = re^{j\theta}$  with  $r > 1$ . Then

$$X(re^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-jn\theta} = 2 \sum_{n=-\infty}^{\infty} x_e(n) s(n) r^{-n} e^{-jn\theta} \quad . \quad (9)$$

But on the unit circle, the  $z$ -transform of  $x_e(n)$  is  $R(e^{j\theta})$  and the  $z$ -transform of the sequence  $s(n)r^{-n}$  is given by  $\frac{1 + r^{-1} z^{-1}}{1 - r^{-1} z^{-1}}$ . Thus, using the complex con-

volution theorem (2), we can rewrite (9) as

$$X(z)|_{z=re^{j\theta}} = \frac{1}{\pi j} \oint \frac{R(z/v)(v+r^{-1})}{v(v-r^{-1})} dv . \quad (10)$$

(In this and subsequent contour integrals, the contour of integration is always taken to be the unit circle).

Equation (10) expresses  $X(z)$  outside the unit circle in terms of its real part on the unit circle. Equation (10) was written as a contour integral to stress the fact that in the physically most interesting case when  $R(z)$  is a rational fraction, evaluation of (10) is most easily performed by contour integration using residues.

Similarly, we may construct  $X(re^{j\theta})$  from  $I(e^{j\theta})$  by noting that  $x(n) = 2x_0(n)s(n) + x(0)\delta(n)$  where  $x_0(n)$  denotes the odd part of  $x(n)$  and  $\delta(n)$  is the unit pulse, defined as unity for  $n = 0$  and zero elsewhere. The result obtained is

$$X(z)|_{z=re^{j\theta}} = \frac{1}{\pi} \oint \frac{I(z/v)(v+r^{-1})}{v(v-r^{-1})} dv + x(0) . \quad (11)$$

Now, Equations (10) and (11) also hold in the limit as  $r \rightarrow 1$ , provided care is taken to evaluate the integral correctly in the presence of a pole on the unit circle. This can be done formally by changing the integrals in (10) and (11) to the Cauchy principal values of these integrals, where the latter is defined as

$$\left. \begin{aligned} \frac{1}{2\pi j} p \oint \frac{f(z)}{z-z_0} dz &= f(z_0) && \text{for } |z_0| < 1 \\ &= 0 && \text{for } |z_0| > 1 \\ &= \frac{1}{2} f(z_0) && \text{for } |z_0| = 1 \end{aligned} \right\} . \quad (12)$$

From (10), (11) and (12), it is a simple matter to construct explicit relations between  $R(e^{j\theta})$  and  $I(e^{j\theta})$ . Alternately, these results could have been derived by appealing directly to Fig. 1, which shows the explicit relations between the real and imaginary part of a causal function to be

$$\left. \begin{aligned} x_0(n) &= \lim_{\alpha \rightarrow 1} x_e(n) w_1(n) \\ x_e(n) &= \lim_{\alpha \rightarrow 1} [x_0(n) w_1(n)] + \delta(n) x(0) \end{aligned} \right\} . \quad (13)$$

Figure 2 shows the ring of convergence for the  $z$ -transform  $W_1(z)$  of  $w_1(n)$  and Fig. 3 shows the poles and zeros of  $W_1(z)$ .

$$jI(z)|_{z=e^{j\theta}} = \frac{1}{2\pi j} p \oint \frac{R(z/v)(v+1)}{v(v-1)} dv . \quad (14)$$

$$R(z)|_{z=e^{j\theta}} = \frac{1}{2\pi} p \oint \frac{I(z/v)(v+1)}{v(v-1)} dv . \quad (15)$$

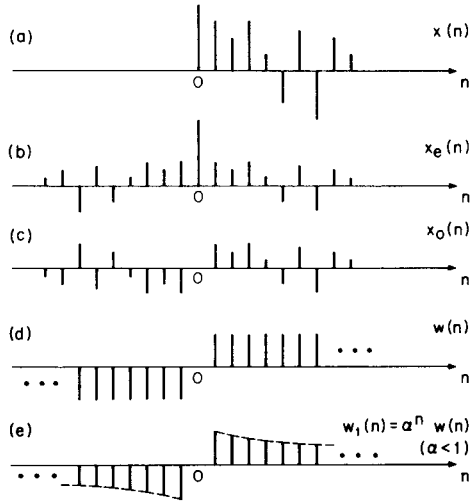


Figure 1.

By setting  $z = e^{j\theta}$  and  $v = e^{j\varphi}$ , we change the contour integrals (14) and (15) to line integrals, yielding

$$\left. \begin{aligned} -I(e^{j\theta}) &= \frac{1}{2\pi} p \int_0^{2\pi} R(e^{j(\varphi-\theta)}) \cot(\varphi/2) d\varphi \\ R(e^{j\theta}) &= \frac{1}{2\pi} p \int_0^{2\pi} I(e^{j(\theta-\varphi)}) \cot(\varphi/2) d\varphi + x(0) \end{aligned} \right\} \quad (16)$$

Similar results can be obtained for the real and imaginary parts of the discrete

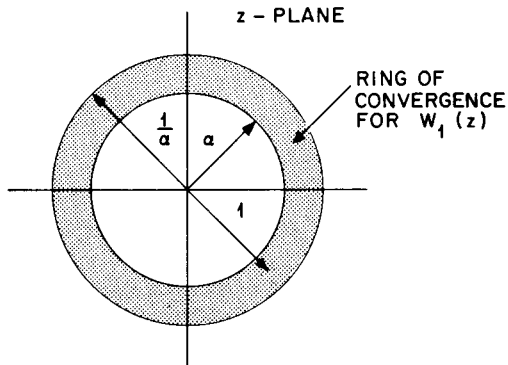
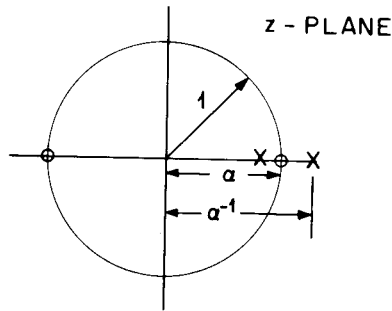


Figure 2.



$$W_1(z) = \frac{z^2 - 1}{(z - a)(z - a^{-1})}$$

Figure 3.

Fourier transform of a finite duration sequence provided that the sequence is causal in the sense that, if the sequence  $x(n)$  is considered to be of duration  $N$ , then  $x(n) = 0$  for  $n > \frac{N}{2}$ . Defining the even and odd parts of  $x(n)$  as

$$x_e(n) = \frac{1}{2} (x[(n)] + x[(-n)]) ,$$

and

$$x_o(n) = \frac{1}{2} (x[(n)] - x[(-n)]) ,$$

it follows (for  $N$  even) that

$$x(n) = x_e(n) f(n) ,$$

and

$$x(n) = x_o(n) f(n) + x(0) \delta(n) + x\left(\frac{N}{2}\right) \delta\left(n - \frac{N}{2}\right) ,$$

where

$$f(n) = \begin{cases} 1 & n = 0, \frac{N}{2} \\ 2 & n = 1, 2, \dots, \frac{N}{2} - 1 \\ 0 & n = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N - 1 \end{cases} .$$

From these relations, we can then derive that the real and imaginary parts,  $R(k)$  and  $I(k)$ , of the DFT of  $x(n)$  are related by

$$jI(k) = \frac{1}{N} \sum_{r=0}^{N-1} R(r) F[(k-r)] , \quad (17)$$

$$R(k) = \frac{1}{N} \sum_{r=0}^{N-1} jI(r) F[(k-r)] + x(0) + (-1)^k x\left(\frac{N}{2}\right). \tag{18}$$

where  $F(k)$  is given by

$$F(k) = \begin{cases} -j \cot \frac{\pi k}{N} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Note that (17) and (18) are circular convolutions which can be numerically evaluated by fast Fourier transform methods. Similar but not identical relations can also be derived if  $N$  is odd.

If instead of working with the  $z$ -transform of a sequence, we choose to work with the logarithm of the  $z$ -transform, then comparable Hilbert transform relations can be derived between the log magnitude of the spectrum and its phase. However, certain theoretical restrictions arise, due to the fact that a) the logarithm of zero diverges and b) the definition of phase is ambiguous. However, the derivative of the phase (with respect to  $z$ ) is not ambiguous; this leads to relationships based on the definition

$$D(z) = \frac{-z}{X(z)} \frac{dX(z)}{dz}. \tag{19}$$

The imaginary part of  $D(e^{j\theta})$  is the derivative of  $\log |D(e^{j\theta})|$  and its real part is the negative of the derivative of the phase of  $D(e^{j\theta})$ . If the inverse  $z$ -transform of  $D(z)$  is causal, relations similar to our previous real-imaginary relations can be derived. For example, results analogous to (16)

$$F = \frac{|X(e^{j\theta})|'}{|X(e^{j\theta})|} = \frac{1}{2\pi} p \int_0^{2\pi} \Psi'(e^{j(\varphi-\theta)}) \cot(\varphi/2) d\varphi \tag{20}$$

$$\Psi'(e^{j\theta}) = -\frac{1}{2\pi} p \int_0^{2\pi} F(e^{j(\varphi-\theta)}) \cot(\varphi/2) d\varphi, \tag{21}$$

where  $|X|$  is the magnitude of the spectrum and  $\Psi$  its phase and the primes denote differentiation with respect to  $\theta$ .

If we impose the condition that  $\Psi$  is an odd function, then it must be zero for  $\varphi = 0$  and (20) and (21) may be integrated to give

$$\log |X(e^{j\theta})| = \frac{1}{2\pi} p \int_0^{2\pi} \Psi(e^{j(\varphi-\theta)}) \cot(\varphi/2) d\varphi, \tag{22}$$

$$\Psi(e^{j\theta}) = -\frac{1}{2\pi} p \int_0^{2\pi} \log |X(e^{j(\varphi-\theta)})| \cot(\varphi/2) d\varphi. \tag{23}$$

The requirement that the inverse  $z$ -transform of  $D(z)$  be zero for  $n < 0$  imposes a restriction on the pole and zero locations of  $X(z)$ . Since the poles of  $D(z)$  occur whenever there are either poles or zeros of  $X(z)$  and since the inverse transform

of  $D(z)$  is zero for  $n < 0$  only if the poles of  $D(z)$  are all within the unit circle, it follows that both poles and zeros of  $X(z)$  must be within the unit circle in order for Equations (20) through (23) to be valid. This is the well-known minimum phase condition [11].

It is also possible to relate the log magnitude and the phase of the DFT by analogous relations provided that the inverse DFT of the logarithm of the DFT is causal. The difficulty in applying this notion is that the logarithm of  $X(k)$  is ambiguous since  $X(k)$  is complex. For the previous case of the  $z$ -transform, this ambiguity was resolved in effect by considering the phase to be a continuous, odd, periodic function; this definition of the phase cannot be applied in this case. Nevertheless, it has been useful computationally for constructing a phase function from the log magnitude of a DFT by computing the inverse DFT of the log magnitude, multiplying by the function  $f(n)$  and then transforming back [4, 5]. The real part of the result is the log magnitude as before and the imaginary part is an approximation to the phase.

#### 4. HILBERT TRANSFORM RELATIONS BETWEEN REAL SIGNALS, AND A FEW APPLICATIONS

The relations of Section 3 were derived via the complex convolution theorem (2) and the requirement of causality. By interchanging time and frequency and using the convolution theorem (1), further relations can be found which are of practical and theoretical interest. One way of obtaining such relations is by the introduction of the "ideal" Hilbert transformer which has a spectrum defined as having the value  $+j$  for  $0 < \varphi < \pi$  and  $-j$  for  $\pi < \varphi < 2\pi$ , or equivalently, a spectrum with flat magnitude vs. frequency and having a phase of  $\pm\pi/2$ . Thus, a Hilbert transformer is a (non-realizable) linear network with this transfer function and, as shown in Fig. 4a, the output of the network is the Hilbert transform of

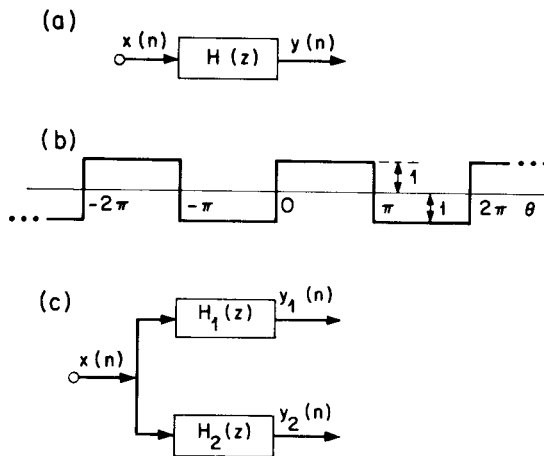


Figure 4.



the input. Hilbert transform relations can also be realized by having two all-pass networks having a phase difference of  $\pi/2$ , as shown in Fig. 4c; such a configuration is useful for synthesis of realizable approximate Hilbert transformers.

The unit pulse response of a Hilbert transformer can be derived by evaluating its inverse  $z$ -transform. Thus

$$\begin{aligned}
 h(n) &= \frac{1}{2\pi j} \oint H(z) z^{n-1} dz , \\
 h(n) &= \frac{1}{2\pi} \left\{ \int_0^\pi j e^{j\theta} d\theta - \int_\pi^{2\pi} j e^{j\theta} d\theta \right\} , \\
 h(n) &= \left. \begin{aligned} &= \frac{1 - e^{j\pi n}}{\pi n} && \text{for } n \neq 0 \\ &= 0 && \text{for } n = 0 \end{aligned} \right\} . \tag{24}
 \end{aligned}$$

From (1) the input-output relations can be written down;

$$y(n) = \frac{1}{\pi} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{x(m) [1 - e^{j\pi(n-m)}]}{(n-m)} . \tag{25}$$

Equation (25) can be inverted by noting that  $X(z) = H^*(z) Y(z)$  (where  $H^*(z)$  is the complex conjugate of  $H(z)$ ); this yields

$$x(n) = -\frac{1}{\pi} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{y(m) [1 - e^{j\pi(n-m)}]}{n-m} . \tag{26}$$

Thus (25) and (26) can be said to be a Hilbert transform signal pair. The graph of  $\frac{1}{\pi} h(n)$  is shown in Figure 5.

The complex signal  $s(n) = x(n) + j y(n)$  (where  $x(n)$  and  $y(n)$  are a Hilbert

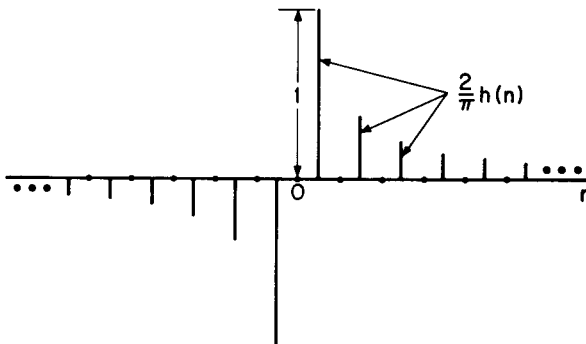


Figure 5.

transform pair) has been called the analytic signal and has the useful property that its spectrum is zero along the bottom half of the unit circle. One application of the analytic signal is to the bandpass sampling problem. Consider the problem of sampling a real signal having the spectrum of Figure 6a. If this signal is passed through the phase splitter of Fig. 4c, the resulting analytic signal has the spectrum shown in Fig. 6b, and thus can be sampled at the rate  $1/B$ . To reconstruct the original signal requires that the samples be applied to the unity gain bandpass filter shown in Figure 6c. The real part of the filtered signal corresponds to the original signal.

Another application of Hilbert transformers is to help create a bandpass spectrum which is arithmetically symmetric about an arbitrary center frequency. Effectively, the ability to do this allows us to design bandpass filters which are linear translations in frequency of prototype low pass filters, thus avoiding the distortions inherent in the standard low pass-bandpass transformations. Figure 7a illustrates a symmetric low pass. When a conventional transformation is applied, the non-symmetric bandpass of Fig. 7b results. Symmetry may be attained with the filter of Fig. 7c; however, we note that the output of such a filter is a sequence of complex numbers and, also, that by merely taking the real part we must introduce the complex conjugate pole, thus destroying the symmetry. To maintain symmetry of a real output over the range 0 through  $\pi$  can be accomplished by the configuration of Fig. 8, where  $H_1(z)$  and  $H_2(z)$  are all pass phase splitters such as shown in Figure 4. If only the real part of the signal is desired then a single phase splitter (rather than two) is needed. A filter satisfying the

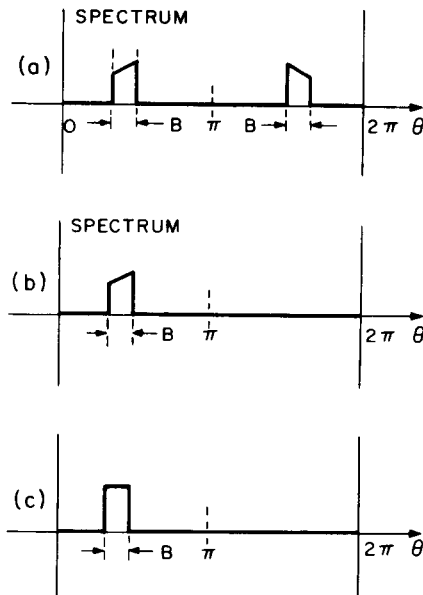


Figure 6.

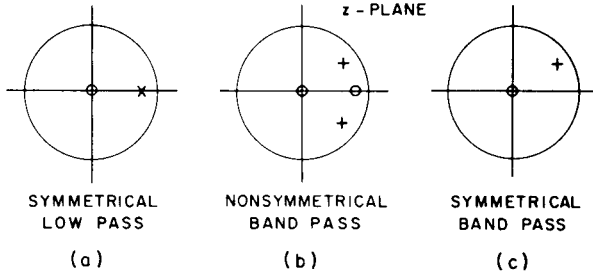


Figure 7.

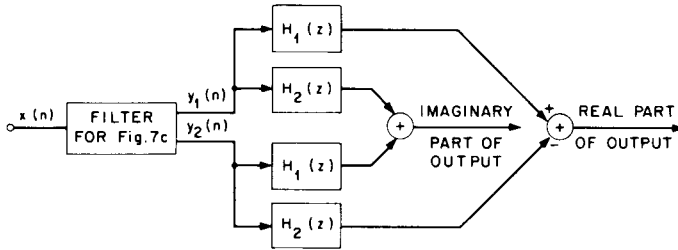


Figure 8.

pole-zero pattern of Fig. 7c is easily made and well-known and has usually been referred to as the coupled form [12, 9].

### 5. DIGITAL HILBERT TRANSFORM NETWORKS

#### A. Recursive Networks

Analog phase splitting networks have been extensively analyzed and synthesized [13, 9]. Since the desired networks are all-pass with constant phase difference over a frequency band, it is feasible to use the bilinear transformation [14, 15, 9] to carry analog designs into the digital domain. The resulting networks are all-pass, so that each pole at, say,  $z = a$ , has a matching zero at  $z = 1/a$ . An equiripple approximation to a constant  $90^\circ$  phase difference is obtained by the use of Jacobian elliptic functions [16], with the added advantage that all the poles and zeros lie on the real axis.

Let the two networks comprising the phase splitter be  $H_1(z)$  and  $H_2(z)$ .

To synthesize the all pass networks  $H_1(z)$  and  $H_2(z)$  in an efficient manner, we note that the first order difference equation

$$y(n) = x(n - 1) + a[y(n - 1) - x(n)] \quad , \quad (27)$$

corresponds to the digital network

$$H(z) = \frac{z^{-1} - a}{1 - az^{-1}} \quad . \quad (28)$$

This shows that an all pass network with a pole at  $z = a$  and a zero at  $z = 1/a$

can be synthesized with a single multiply in a first order difference equation. In Fig. 9 a complete digital 90° phase splitter is shown which meets the requirements that the phase difference deviates from 90° in an equiripple manner by ±1° in the range 10° through 120° along the unit circle. From (28) we see that the coefficients in Fig. 9 are equal to the pole positions. The nomenclature of Fig. 9 is the following: the box  $z^{-1}$  signifies a unit delay, the plus signifies addition and a number-arrow combination signifies both direction of data flow and multiplication by the number. Arrows without numbers signify only data flow.

Now it is well known [9, 12, 17, 18, 19, 20, 21, 22] that because of finite register length, the performance of the actual filter deviates somewhat from that of the design. These effects can be categorized as follows:

- a) Quantization of the input signal
- b) Roundoff noise caused by the multiplications (fixed point arithmetic is assumed)
- c) Deadband effect
- d) A fixed deviation in the filter characteristic caused by inexact coefficients.

The analysis of these effects is simplified because the networks  $H_1(z)$  and  $H_2(z)$  are all-pass. Thus, signal to noise ratios caused by (a) are the same at the output of the networks as at the input. Item (b) can be analyzed for  $H_1(z)$  or  $H_2(z)$  by inserting noise generators at all adder nodes following multiplications. But each noise is then filtered by a cascade combination of the pole of the section in which the noise is generated, and an all-pass network. The well-known formula for the output variance of a network which has been subjected to a white noise input with uniform probability density of amplitude is given by

$$(E_0^2 / 12) \sum_{n=0}^{\infty} h^2(n) ,$$

where  $E_0$  is a quantum step and  $h(n)$  is the network impulse response. For a single pole at  $z = a$ ,  $h(n) = a^n$ ; assuming  $m$  independent noise generators and  $m$  poles at  $a_1, a_2, \dots, a_m$  causes a total variance

$$\sigma^2 = E_0^2 / 12 \sum_{i=1}^m \frac{1}{1 - a_i^2} . \tag{29}$$

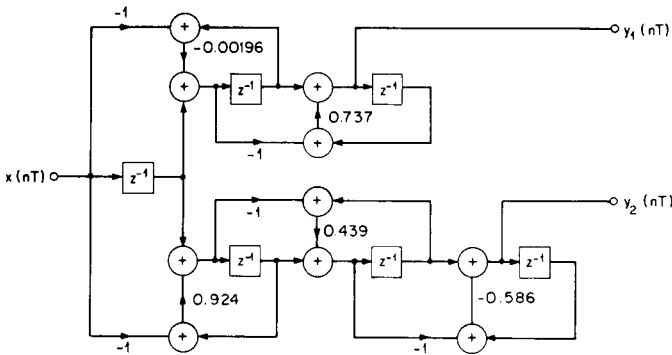


Figure 9.

We see that only values of  $a_i$  near unity cause much noise. Thus, for our numerical design example, only about 1 bit of noise is generated. Item (c) can be analyzed by similar considerations but it is probably not important for band-pass phase splitters anyway, since it is only an effect when the input is a constant.

For small errors in coefficients, item (d) can be analyzed in a manner similar to that of reference [12]. The realization chosen in Fig. 9 guarantees that even though a given coefficient is in error, the poles and zeros of the networks remain reciprocals so that only the phase response of the network can be effected. Let the phase response due to a pole zero pair at  $a, 1/a$  be  $\Psi(\varphi, a)$ . The phase error for a coefficient error  $\Delta a$  is approximated by

$$\Delta\Psi \approx \frac{\partial\Psi}{\partial a} \Delta a + \frac{1}{2} \frac{\partial^2\Psi}{\partial a^2} (\Delta a)^2 + \dots$$

It is easily shown, using (28), that

$$\Psi = 2 \tan^{-1} \left( \frac{\sin \varphi}{a - \cos \varphi} \right);$$

$$\frac{\partial\Psi}{\partial a} = \frac{-2 \sin \varphi}{a^2 - 2a \cos \varphi + 1},$$

$$\frac{1}{2} \frac{\partial^2\Psi}{\partial a^2} = \frac{2 \sin \varphi (a - \cos \varphi)}{(a^2 - 2a \cos \varphi + 1)^2}.$$

Therefore a good approximation to the error in phase, is given by the early terms of the series

$$\Delta\Psi \approx \frac{-2 \sin \varphi}{a^2 - 2a \cos \varphi + 1} \Delta a + \frac{2 \sin \varphi (a - \cos \varphi)}{(a^2 - 2a \cos \varphi + 1)^2} (\Delta a)^2 + \dots$$

Using this approximation for each of the poles one can estimate how many bits are necessary to keep the phase error within a given tolerance. Of course, once a coefficient has been specified, the phase difference can be computed precisely.

## B. Non-Recursive Networks

The phase splitting network we have described above is called recursive because, in the computation (27), a new output depends on a previous output. Recursive networks always have poles and unit pulse response of infinite duration. By contrast, a nonrecursive network has only zeros and a finite duration unit pulse response. As we have shown, perfectly all pass, recursive phase splitters with equiripple phase characteristics can be constructed. Such criteria cannot, in general, be met by nonrecursive networks. However, useful constructions are certainly possible. We have studied one method of nonrecursive design which is based on sampling in the frequency domain [19]. The sampling formula which relates the  $z$ -transform of a finite duration sequence of length  $N$  to the values  $H_k$  of its DFT is given by

$$H(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H_k}{1 - z^{-1} W^k}, \quad W = e^{-j\frac{2\pi}{N}}. \quad (30)$$

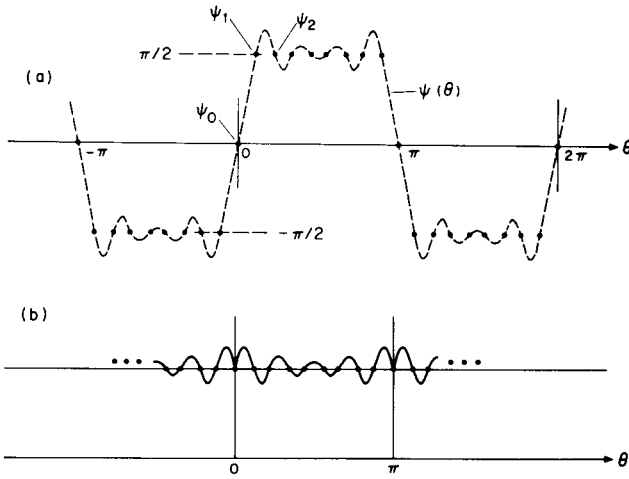


Figure 10.

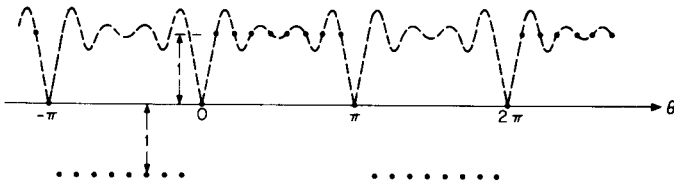


Figure 11.

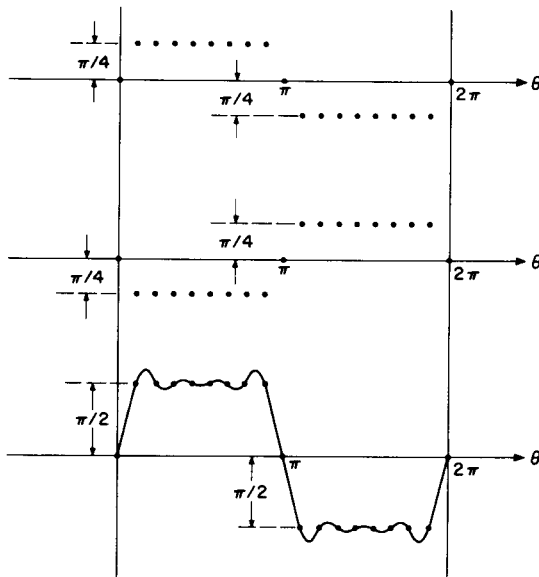


Figure 12.

Thus, an approach to the design of a nonrecursive Hilbert transformer is to specify the phase and magnitude of the network at the  $N$  equally spaced points around the unit circle, that is, the  $N$  values of  $H_k$ . If the unit pulse response of  $H(z)$  is constrained to be real, then we must specify that  $H_k$  has even magnitude and odd phase. For example, if we specify the magnitude of  $H_k$  to be unity for all  $k$  and the phase to be  $\pi/2$  in the first half of the period and  $-\pi/2$  in the second half, as in Fig. 10, then an interpolated spectrum results, such as shown by the dotted curve of Figure 10.

Exact  $90^\circ$  phase can be attained at every frequency, by specifying that the  $H_k$  be purely imaginary. However, for real unit pulse response, such a phase shifter must have a magnitude characteristic which passes through zero at  $0, \pi, 2\pi$  etc., as shown in Figure 11. Thus, an ideal phase can be attained by further degrading the all pass property of the network near  $0$  and  $\pi$ .

Nonrecursive networks can also be arranged as phase splitters, which has the advantage that the two components of the resulting analytic signal can have the same delay. It has been found experimentally that the arrangement indicated in Fig. 12, whereby each arm of the splitter has nominal phase of  $\pm\pi/4$  leads to a low value of ripple of the interpolated phase difference. Further reduction of ripple is possible at the expense of bandwidth, by specifying intermediate values of  $H_k$  in the transition region when the sign of the phase is changing.

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