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Generalized Superposition*

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The primary means for the analysis and synthesis of linear systems stems from their defining property, the principle of superposition. In this paper a generalization of this principle, which provides a classification for a variety of nonlinear systems, is discussed. Systems within each class are shown to differ only by a linear system. The application of this approach to problems in nonlinear filtering is proposed.

I. INTRODUCTION

In many cases nonlinear systems can be adequately treated by collecting together those that share common properties; i.e., by classifying nonlinear systems and exploiting the properties common to each class. In this paper, one such approach to the characterization of a broad class of nonlinear systems is proposed. This approach suggests a classification of many nonlinear systems in such a way that each class is defined by a principle of superposition which is similar to the principle of superposition for linear systems. Each class has a canonic representation that contains a linear system, and systems within a class differ only in the linear portion of this representation.

The classification of systems based on this generalized principle of superposition suggests an approach to nonlinear filtering of signals which have been nonlinearly combined. In this approach signals to be separated are considered part of a vector space with vector addition taken to be the same operation as that under which the signals were combined. The class of nonlinear filters used then correspond to linear transformations on this vector space.

The mathematical vehicle that has been used for the development of

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the theory is linear algebra. Much of the formalism required for a thorough treatment is omitted from the present discussion under the assumption that a somewhat heuristic treatment will be more readable than a formal derivation of results. A more detailed treatment has been given elsewhere [Oppenheim, (1965)[a]].

II. GENERALIZED SUPERPOSITION AND HOMOMORPHIC SYSTEMS

The principle of superposition, as it is stated for linear systems, requires that if T is the system transformation, then for any two inputs $x_1(t)$ and $x_2(t)$ and any scalar c,

$$T[x_1(t) + x_2(t)] = T[x_1(t)] + T[x_2(t)] \tag{1}$$

and

$$T[cx_1(t)] = cT[x_1(t)]. \tag{2}$$

From this definition it is clear that a system with transformation ψ given by

$$\psi[x(t)] = [x(t)]^2, \tag{3}$$

is nonlinear. However,

$$\psi[x_1(t)x_2(t)] = \psi[x_1(t)]\psi[x_2(t)] \tag{4}$$

and

$$\psi[x_1^c(t)] = [\psi[x_1(t)]]^c. \tag{5}$$

The transformation of Eq. (3) can be said to satisfy a principle of superposition in the sense that its response to a product of inputs is the product of the individual responses. This suggests, then, a generalization of the principle of superposition, as is stated for linear systems, which will encompass at least some nonlinear systems. To state this principle formally, let us consider a system with transformation ϕ, and let $\{x(t)\}$ denote the set of possible inputs and $\{y(t)\}$ denote the set of possible outputs. Let

$$x_1(t) \bigcirc x_2(t)$$

denote the combination of any two inputs under an operation $\bigcirc$ (e.g. addition, multiplication, convolution, etc.) and let $y_1(t) \square y_2(t)$ denote the combination of any two outputs under an operation $\square$. Similarly, let $c * x(t)$ denote the combination of an input $x(t)$ with a scalar c and

$c : y(t)$ denote the combination of an output $y(t)$ with a scalar c. Then the system can be said to satisfy a generalized principle of superposition if

$$\phi[x_1(t) \bigcirc x_2(t)] = \phi[x_1(t)] \square \phi[x_2(t)] \tag{6}$$

and

$$\phi[c * x(t)] = c : \phi[x(t)] \tag{7}$$

When the operations $\bigcirc$ and $\square$ correspond to addition and the operations $*$ and $:$ correspond to multiplication, the system ϕ will, of course, be a linear system.

If the set of system inputs is such that it can be represented as a vector space with vector addition corresponding to the combination of two inputs under the operation $\bigcirc$ and scalar multiplication corresponding to the combination of the inputs with scalars under the operation $*$, then the system transformation can be represented by a linear transformation between vector spaces. We should also require, of course, that the set of outputs be representable by a vector space, with the operations $\square$ and $:$ corresponding to vector addition and scaler multiplication, respectively. However, this is guaranteed if the inputs constitute a vector space and the system transformation has the properties specified by Eq. (6) and (7).

Examples of simple system transformations, that are representable as linear transformations between vector spaces for vector addition taken as multiplication in both the input and output vector spaces, are

1. $\phi[x(t)] = [x(t)]^n$ (power-law devices)
2. $\phi[x(t)] = \text{sign}[x(t)]$ (infinite clipper)
3. $\phi[x(t)] = |x(t)|$ (full-wave rectifier)

Although in each of these simple cases the systems are memoryless, it will be apparent, when the canonic representation of these systems is discussed, that this is not a general restriction.

The generalization stated in Eq. (6) and (7) does not require the additional constraint that the set of inputs constitute a vector space, and indeed we can imagine choices for the operations $\bigcirc$ and $\square$ which do not satisfy the algebraic postulates of vector addition. In this paper, however, only those cases for which these postulates are satisfied, will be discussed. This permits a direct application of the theorems of linear algebra to the characterization of these systems. Such systems, which

can be represented as linear transformations between vector spaces, will be referred to as homomorphic systems, a term suggested by the algebraic definition of a homomorphic (i.e., linear) mapping between vector spaces. The operation $\bigcirc$ will be referred to as the input operation of the system, and the operation $\square$ will be referred to as the output operation. A homomorphic system with input operation $\bigcirc$, output operation $\square$, and system transformation ϕ will be represented as shown in Fig. 1. (Strictly speaking, the input and output operations do not specify completely the interpretation of scalar multiplication for the input and output vector spaces. The operations $*$ and : however, are inferred by the operations $\bigcirc$ and $\square$ for scalars that are rational, and in many cases are suggested in general.)

To investigate the generality of the class of homomorphic systems, let us consider a system with a transformation ϕ. Let the inputs $\{x(t)\}$ to this system constitute a vector space with $\bigcirc$ as vector addition and $*$ as scalar multiplication. Then

1. there is, at most, one choice for the operations $\square$ and : , so that the system is homomorphic with $\bigcirc$ and $*$ as the input operations.
2. if ϕ is invertible, so that there is a one to one correspondence between inputs and outputs, there is at least one choice for the operations $\square$ and : , so that the system is homomorphic with $\bigcirc$ and $\diamondsuit$ as the input operations; i.e., all invertible systems are homomorphic.

The first statement follows in a straightforward way from Eqs. (6) and (7). Specifically, let y_1 and y_2 represent any two system outputs and let x_1 and x_2 represent any inputs which produce these outputs so that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Then Eq. (6) requires that $y_1 \square y_2$ be the output produced by input $x_1 \bigcirc x_2$. Since we assume that the system is well defined, $y_1 \square y_2$ is unique. Similarly, from Eq. (7), $c : y_1$ is unique.

The second statement follows by demonstrating that the input opera-

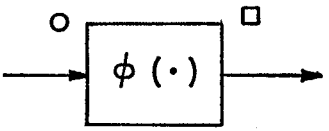

FIG. 1. Representation of a homomorphic system with input operation $\bigcirc$, output operation $\square$ and system transformation ϕ.

tions and the invertibility of the system induce an appropriate set of output operations. Let us define $y_1 \square y_2$ and $c : y_1$ as

$$y_1 \square y_2 \equiv \phi[\phi^{-1}(y_1) \bigcirc \phi^{-1}(y_2)] \tag{8}$$

and

$$c : y_1 \equiv \phi[c * \phi^{-1}(y_1)] \tag{9}$$

The output operations given by Eqs. (8) and (9) satisfy Eqs. (6) and (7). Furthermore, from statement 1 we are guaranteed that the output operations given by Eqs. (8) and (9) are the only choice for these operations. This is not to suggest, of course, that we shall always want to determine the output operations for a specified system by applying Eqs. (8) and (9), but we are at least assured that, however these operations are determined, they will always be the same.

III. REPRESENTATION OF HOMOMORPHIC SYSTEMS

In considering any class of homomorphic systems defined by specified input and output vector spaces, the question naturally arises as to how to exploit the principle of superposition which is the defining property for the class. Since the systems correspond to algebraically linear transformations between vector spaces, their representation is no different than that used for linear systems. Specifically, the systems may be represented by the mapping of the basis vectors from the input space to the output space. An alternative point of view is that both the input space and the output space are isomorphic with spaces of the same dimensions for which vector addition corresponds to the sum of functions, and scalar multiplication to the product of the scalars and the functions. The system transformation is then represented by a linear transformation between these new vector spaces. To state this formally, it is convenient to restrict the input and output vector spaces to be separable Hilbert spaces. In essence, this requires that their dimension be countable, that an inner product can be defined on the space, and that the space possesses an orthonormal basis. Let the set of functions $\{\ell\}$ be a separable Hilbert space of the same dimension as the input vector space with addition and scalar multiplication given by

$$\ell_1 \bigcirc \ell_2 = \ell_1 + \ell_2$$

and

$$c * \ell = c\ell.$$

The vector spaces $\{x\}$ and $\{\ell\}$ are isomorphic and, consequently, we can find an invertibile homomorphic system which we denote as $\alpha_{\bigcirc}$, for which $\{x\}$ is the set of inputs and $\{\ell\}$ is the set of outputs. Thus $\alpha_{\bigcirc}$ is invertible and has input operation $\bigcirc$ and output operation $+$. In a similar manner, there exists an invertible homomorphic system, denoted by $\beta_{\square}$ having input operation $\square$ and output operation $+$.

Since $\alpha_{\bigcirc}$ and $\beta_{\square}$ are both invertible, the system ϕ can then be represented as shown in Fig. 2. The system enclosed by dashed lines is a linear system, that is, it is a homomorphic system with addition as both the input and output operations. If L_ϕ denotes this linear transformation, then Fig. 2 can be redrawn as shown in Fig. 3. This cascade will be referred to as the canonic form for homomorphic systems. It is important to note that the system $\alpha_{\bigcirc}$ is determined only by the set of inputs and the input operations, and that the system $\beta_{\square}$ (or $\beta_{\square}^{-1}$) is determined only by the set of outputs and the output operations. If we consider classifying homomorphic systems by their input and output spaces (including a specification of the input and output operations), then the systems $\alpha_{\bigcirc}$ and $\beta_{\square}$ are characteristic of a class. Consequently, homomorphic systems within a class differ only in the linear portion of the canonic representation for that class. The system of Fig. 3 is homomorphic with input operation $\bigcirc$ and output operation $\square$ for any choice for the linear system L_ϕ. Consequently, when the characteristic systems $\alpha_{\bigcirc}$ and $\beta_{\square}$ for a class are known, the class of homomorphic systems can be generated by varying the linear system L_ϕ. It can also be shown that if and only if the input and output operations for a class are memoryless (such as addition and

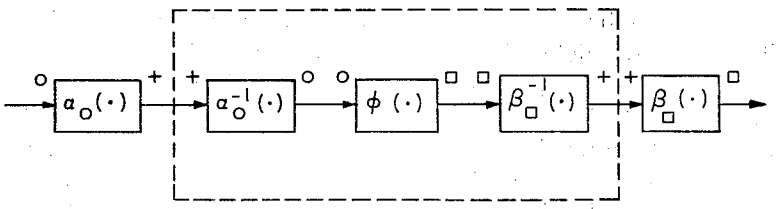

FIG. 2. Equivalent representation of homomorphic systems

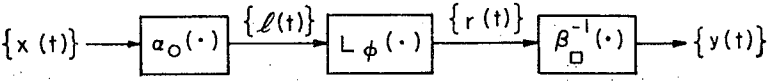

FIG. 3. Canonic representation of homomorphic systems

multiplication in contrast with convolution), then the characteristic systems for that class can be chosen to be memoryless. In these cases then, all of the system memory is concentrated in the linear portion of the canonic representation.

IV. GENERALIZED LINEAR FILTERING

The linear filtering problem, as it is often stated, is concerned with the use of a linear system for the recovery of a signal after it has been added to noise. From a vector-space point of view, the linear filtering problem can be considered as that of determining a linear transformation on a vector space such that the length or norm of the error vector is minimum. The norm associated with the vector space specifies the error criterion to be used.

From the previous discussion it should be clear that a generalization can be carried out for the filtering of signal and noise that have been nonadditively combined, provided that the rule of combination satisfies the algebraic postulates of vector addition. For example, if we wish to recover a signal $s(t)$ after it has been combined with noise $n(t)$ such that the received signal is $s(t) \bigcirc n(t)$, we may associate $s(t)$ and $n(t)$with vectors in a vector space and the operation $\bigcirc$ with vector addition. The class of linear transformations on this vector space would then be associated with the class of homomorphic systems having the operation $\bigcirc$ as both the input operation and the output operation. Hence, in generalizing the linear filtering problem to homomorphic filtering, the class of filters from which the optimum is to be selected will be that class of homomorphic systems having input and output operations that are identical to the rule under which the signals that are to be separated have been combined. With this restriction on the class of filters, it follows that the determination of the optimum filter reduces to the determination of an optimum linear filter. Specifically, let x_1 and x_2 denote two signals that have been combined under the operation $\bigcirc$. Then the canonic form for the class of homomorphic filters which will be used to recover x_1 or x_2 is depicted in Fig. 4, where $\alpha_{\bigcirc}$ and its inverse are characteristic of the class. Consequently, the choice of system from this class rests only on the choice of the linear system L. But, since $\alpha_{\bigcirc}$ is homomorphic, the input to the linear system is

$$\ell = \alpha_{\bigcirc}[x_1] + \alpha_{\bigcirc}[x_2] = \ell_1 + \ell_2 .$$

If we wish to use the homomorphic system to recover x_1 for example, then the desired output of the over-all system is x_1. Consequently, the desired output from the linear system is ℓ_1. Thus, we wish to select the

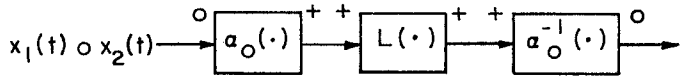

FIG. 4. Canonic representation of homomorphic filters

linear system L, so that with input $\ell_1 + \ell_2$ the output is closest in some sense to the desired output y_1. This, of course, is just the statement of the linear-filtering problem, with the exception that we have not yet specified an error criterion under which to carry out the optimization. Since the linear system is all that needs to be determined to obtain the homomorphic filter, it seems reasonable to suppose that we may just determine the linear system using an error criterion normally used for linear filtering problems; e.g., mean-square or integral-square error. The formalism for showing this has been described elsewhere [Oppenheim, (1965)[b]]. In brief, what is required is to show that a norm can be selected for the vector space associated with the system outputs, such that the norm of the error vector is minimum, if and only if the norm of the error vector associated with the vector space of outputs of the linear system is minimum. But, if two vector spaces are isomorphic, then a norm in one can induce a norm in the other. In other words, if y represents any output of a homomorphic system, then one can choose as a norm on the output space

$$\| y \| \equiv \| \beta_\square(y) \|.$$

With this choice as the norm on the set of outputs, the error measurement will be numerically equal before and after the transformation $\beta_\square^{-1}$. The conclusion is that mean-square or integral-square error at the output of the linear system is a meaningful error measurement for the over-all system.

The notion of generalized linear filtering has found immediate application for the filtering of multiplied signals and the filtering of convolved signals. In this case the characteristic system $\alpha_\circ$ has as its output the logarithm of the input. Since signals in a vector space under multiplication can, in general, be positive, negative, or complex (but not zero), the output of the logarithmic transformation will, in general, be complex.

In the filtering of convolved signals [Oppenheim, (1966)] the characteristic system $A_\otimes$ is defined by the property that

$$A_\otimes[s_1(t) \otimes s_2(t)] = A_\otimes[s_1(t)] + A_\otimes[s_2(t)],$$

where $\otimes$ denotes convolution.

By taking advantage of the fact that the convolution of $s_1(t)$ and $s_2(t)$ has as its Fourier transform the product of the transforms of $s_1(t)$ and $s_2(t)$, one realization of the system $A_\otimes$ is obtained by defining it by the property

$$F[A_\otimes(s(t))] = \log [F(s(t))], \tag{15}$$

where F denotes Fourier transformation.

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