FILTER BANK INTERPOLATION AND RECONSTRUCTION FROM GENERALIZED AND RECURRENT NONUNIFORM SAMPLES

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ABSTRACT

This paper introduces a filter bank interpretation of various sampling strategies, which leads to efficient interpolation and reconstruction methods. An identity, referred to as the Interpolation Identity, is used to obtain particularly efficient discrete-time (DT) systems for interpolation to uniform Nyquist samples, either for further processing in that form or for conversion to continuous-time (CT). The Interpolation Identity also leads to a new class of sampling theorems including an extension of Papoulis' generalized sampling expansion.

1. INTRODUCTION

Discrete-time signal processing (DSP) inherently relies on sampling a continuous time signal to obtain a DT representation of the signal. The most common form of sampling used in the context of DSP is uniform (periodic) sampling. However, there are a variety of applications in which data is sampled in other ways, such as nonuniformly in time or through multichannel data acquisition. There are also applications where we can benefit from deliberately introducing more elaborate sampling schemes.

Several extensions of the uniform sampling theorem are well known. Specifically, it is well established that a bandlimited signal can be uniquely determined from nonuniform samples, provided that the average sampling rate exceeds the Nyquist rate ([7]). However, in contrast to uniform sampling, reconstruction of the CT signal from nonuniform samples using the direct interpolation procedure is computationally difficult. Several alternative reconstruction methods from nonuniform samples have been previously suggested. These methods involve iterative algorithms (e.g. [1]) which are computationally demanding and have potential issues of convergence. Another well known sampling theorem by Papoulis ([5]), which generalizes uniform sampling of a signal, states that a bandlimited signal can be reconstructed from uniformly spaced samples of the outputs of M linear time-invariant (LTI) systems with the signal as their input, sampled at one-M'th of the Nyquist rate. However, the reconstruction from these generalized samples is again computationally complex. In order to exploit alternative sampling methods in various applications practical, efficient reconstruction algorithms are required.

In this paper we formulate an identity that leads to efficient reconstruction methods from generalized samples, as well as efficient interpolation to uniformly spaced samples. We then develop a new non-iterative approach to reconstruction from recurrent nonuniform samples. The resulting procedure consists of processing the samples with a bank of LTI filters either to reconstruct the original bandlimited CT signal or to interpolate the recurrent nonuniform samples to uniformly spaced samples. In addition to offering efficient implementations, the filter bank framework leads to a new class of sampling theorems. As an example, we show that applying the identity to perfect reconstruction filter banks results in a generalization of Papoulis' sampling theorem ([5]).

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2. THE INTERPOLATION IDENTITY

Throughout the paper we use the variables Ω and ω to denote frequency variables for CT and DT respectively. Capital letters are used to denote the Fourier transform. Parentheses are used for CT signals, and brackets for DT signals. We assume that all signals are bandlimited to W, i.e. their Fourier transform is zero for $W \leq |\Omega|$. T_Q denotes the Nyquist period given by $T_Q = \pi/W$. We use the notation depicted in Fig. 1 to denote conversion of a sequence of samples $x_c(nT)$ to a CT signal $y_c(t)$ where $y_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$.

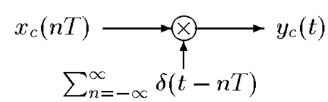

Figure 1: Converting samples $x_c(nT)$ to a CT signal $y_c(t)$.

The following equivalence, which we refer to as the Interpolation Identity, will be used in subsequent sections to arrive at efficient implementations of the reconstruction from generalized and nonuniformly spaced samples. The proof of a more general form of this identity is given in [2].

The Interpolation Identity: *for any CT signal $x_c(t)$ and impulse response $h(t)$ with corresponding frequency response $H(\Omega)$ bandlimited to $W = \pi/T_Q$ the block diagrams depicted in Fig. 2(a) and 2(b) with*

$$\widetilde{H}(\omega) = \frac{1}{T_Q} H\left(\frac{\omega}{T_Q}\right), \quad |\omega| \leq \pi \tag{1}$$

and $T = NT_Q$ are equivalent.

The block diagram of Fig. 2(a) consists of converting a sequence of samples to a CT signal followed by CT filtering. The block diagram of Fig. 2(b) consists of expanding the sequence of samples by a factor of N. The expanded output is then filtered by a DT filter with frequency response given by (1), followed by impulse modulation and CT low-pass filtering.

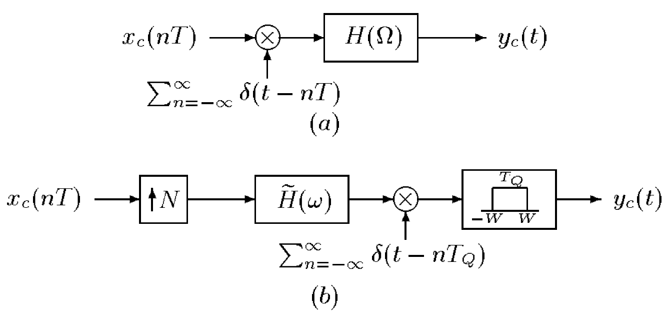

Figure 2: The Interpolation Identity.

Note, that (1) implies that $\tilde{h}[n] = h(nT_Q)$ where $\tilde{h}[n]$ is the DT impulse response associated with $\widetilde{H}(\omega)$. Since $T = NT_Q$, the sequence $x_c(nT)$ is in general an undersampled representation of $x_c(t)$ and consequently $y_c(t)$ is in general an aliased reconstruction from these samples.

3. INTERPOLATION AND RECONSTRUCTION FROM SAMPLES OF A SIGNAL AND ITS DERIVATIVE

As an example of the application of the Interpolation Identity consider sampling a signal and its derivative. It is well known that a signal can be recovered from samples of the signal and its derivative at half the Nyquist rate ([3]) using the reconstruction formula

$$x_c(t) = \sum_{k=-\infty}^{\infty} \mathrm{sinc}^2(\pi(t-k2T_Q)/2T_Q)(f[k] + (t-k2T_Q)f'[k]) \tag{2}$$

where $f[n] = x_c(n2T_Q)$ and $f'[n] = \frac{dx_c(t)}{dt}\big|_{t=n2T_Q}$. Note that the sequences $f[n]$ and $f'[n]$ are undersampled representations of $x_c(t)$ and $\frac{dx_c(t)}{dt}$ respectively.

Eq. (2) can be implemented using the CT filter bank (CTFB) depicted in Fig. 3, with $h_0(t) = \mathrm{sinc}^2(\pi t/2T_Q)$ and $h_1(t) = t\mathrm{sinc}^2(\pi t/2T_Q)$. If instead of reconstructing $x_c(t)$ we are interested in converting $f[n]$ and $f'[n]$ to uniform Nyquist samples of $x_c(t)$, the interpolation formula is obtained by substituting $t = nT_Q$ in (2).

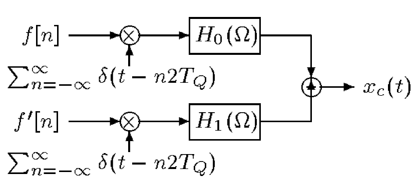

Figure 3: Reconstruction from samples of a signal and its derivative at half the Nyquist rate.

Interpolation and reconstruction of $x_c(t)$ using (2) are difficult to implement directly. However, both interpolation and reconstruction can be implemented in a simpler form by applying the Interpolation Identity to the system in Fig. 3. Specifically, the CTFB of Fig. 3 can be converted to a DT filter bank (DTFB) followed by a CT low-pass filter (LPF). Applying the equivalence of Fig. 2 to each branch in Fig. 3 and moving the identical impulse train modulation and LPF in each branch outside the summer we obtain the equivalent implementation in Fig. 4 where $\tilde{h}_p[n] = h_p(nT_Q)$ for $p = 1, 2$. As with the CTFB, the overall output of Fig. 4 is the original CT signal $x_c(t)$. Since $x_c(t)$ is reconstructed through low-pass filtering of a uniformly spaced impulse train, the impulse train values $x[n]$ must correspond to uniformly spaced samples of $x_c(t)$ at the Nyquist rate. Thus, the DTFB provides a DT mechanism for converting the generalized samples to uniform Nyquist samples.

Using the DTFB of Fig. 4 we can reconstruct the CT signal very efficiently from samples of the signal and its derivative by exploiting the many known results regarding efficient implementation of the filters comprising a DTFB (see e.g. [6]). By following an analogous procedure, we can arrive at efficient interpolation and reconstruction methods for other forms of generalized samples. In the next section we focus on efficient implementation of the reconstruction from recurrent nonuniform samples using a bank of CT and DT filters.

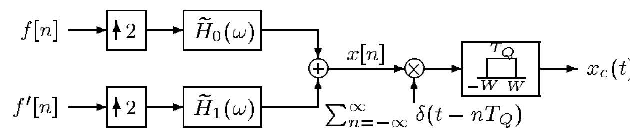

Figure 4: Interpolation and reconstruction using a DTFB.

4. RECURRENT NONUNIFORM SAMPLING

It is well established that a CT signal $x_c(t)$ can be reconstructed from its samples at a set of sampling times $\{t_n\}$ if the average sampling period is smaller than the Nyquist period, where the average sampling period is defined as $\lim_{n\to\infty} \frac{t_n}{n}$ ([7]). In this section we focus on an efficient implementation of the reconstruction for the case of recurrent nonuniform sampling. In this form of sampling the sampling points are divided into groups of N points each. The groups have a recurrent period, denoted by T, equal to N times the Nyquist period T_Q. Each period consists of N nonuniform sampling points. Denoting the points in one period by t_p $p = 0, 1, \ldots N-1$, the complete set of sampling points are

$$t_p + nT \quad p = 0, 1, \ldots N-1, \; n \in (-\infty, \infty) \tag{3}$$

where $T = NT_Q$. Without loss of generality we will assume throughout that $t_0 = 0$.

The reconstruction formula is given by (see [2] and [8]):

$$x_c(t) = \sum_{n=-\infty}^{\infty} \sum_{p=0}^{N-1} x_c(nT + t_p) \frac{a_p(-1)^{nN} \prod_{q=0}^{N-1} \sin(\pi(t-t_q)/T)}{t - nT - t_p} \tag{4}$$

where

$$a_p = \frac{1}{\prod_{q=0, q\neq p}^{N-1} \sin(\pi(t_p - t_q)/T)}. \tag{5}$$

Reconstruction from recurrent nonuniform samples using (4) is considerably more complex than reconstruction from uniform samples. Eq. (4) has a time varying form and therefore cannot be implemented directly using LTI filters. If implemented directly the infinite sum in (4) must be approximated in the time domain, resulting in relatively large approximation errors. In the next subsection, we develop an efficient implementation of (4) through the use of a bank of CT LTI filters. In section 4.2 we develop an alternative implementation using a bank of DT LTI filters.

4.1. Reconstruction From Recurrent Nonuniform Samples Using A Continuous-Time Filter Bank

In this sub-section we develop a CTFB representation of Eq. (4). To this end, we interchange the order of summations in (4), and denote the inner sum by $f_p(t)$, i.e.

$$f_p(t) = \sum_{k=-\infty}^{\infty} x_c(kT+t_p) \frac{a_p(-1)^{kN} \prod_{q=0}^{N-1} \sin(\pi(t-t_q)/T)}{t-kT-t_p}. \tag{6}$$

Using the relation $\sin(t-k\pi) = (-1)^k \sin(t)$ we rewrite $f_p(t)$ as

$$f_p(t) = \sum_{k=-\infty}^{\infty} x_c(kT+t_p) \frac{a_p \prod_{q=0}^{N-1} \sin(\pi(t-t_q-kT)/T)}{t-kT-t_p}. \tag{7}$$

Eq. (7) can be expressed as a convolution. Specifically,

$$f_p(t) = s_p(t) * h_p(t) \tag{8}$$

with

$$h_p(t) = a_p \frac{\prod_{q=0}^{N-1} \sin(\pi(t+t_p-t_q)/T)}{t} \tag{9}$$

and $s_p(t)$ is an impulse train of samples, i.e.

$$s_p(t) = \sum_{k=-\infty}^{\infty} x_c(kT+t_p)\delta(t-kT-t_p). \tag{10}$$

Using (8) we can express (4) as a sum of N convolutions:

$$x_c(t) = \sum_{p=0}^{N-1} s_p(t) * h_p(t). \tag{11}$$

Eq. (11) can be interpreted as a CTFB as depicted in Fig . 5. The signals $s_p(t)$ are formed according to (10), i.e. the samples are divided into N sub-sequences, where each sub-sequence corresponds to samples at one-N'th of the Nyquist rate of a time-shifted version of the original signal. Each sub-sequence is converted to a CT signal $s_p(t)$ using a shifted impulse train. The signal $s_p(t)$ is then filtered by a CT filter with impulse response $h_p(t)$ given by (9). Summing the outputs of the N branches results in the reconstructed signal $x_c(t)$. Note, that each one of the sub-sequences corresponds to uniform samples *at one-N'th of the Nyquist rate*. Therefore, the output of each branch of the filter bank is an aliased and filtered version of $x_c(t)$. The filters as specified by (9) have the inherent property that the aliasing components of the filter outputs cancel in forming the summed output $x_c(t)$. The filters have the additional properties that $H_p(\Omega) = 0$ for $|\Omega| > W$, i.e. they are bandlimited to the same bandwidth as the CT signal and each frequency response $H_p(\Omega)$ is piecewise constant over frequency intervals of length $2W/N$.

4.2. Interpolation and Reconstruction From Recurrent Nonuniform Samples Using A Discrete-Time Filter Bank

Following an analogous procedure to section 3, the CTFB in Fig. 5 can be converted to a DTFB followed by a CT LPF. Noting that the delay of t_p in the impulse train of the p'th branch in Fig. 5 can be incorporated into the filter $h_p(t)$ and applying the Interpolation

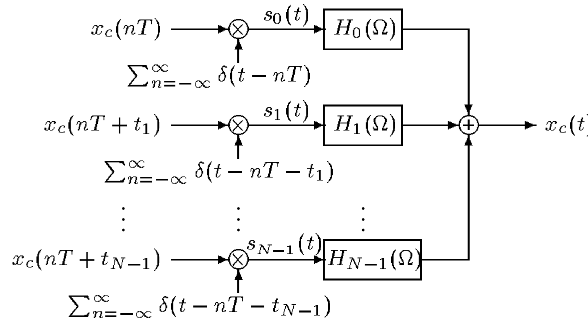

Figure 5: Reconstruction from recurrent nonuniform samples using a CTFB.

Identity of Fig. 2 to each resulting branch we obtain the equivalent implementation in Fig. 6 where

$$\widetilde{H}_p(\omega) = \frac{1}{T_Q} H_p\left(\frac{\omega}{T_Q}\right) e^{-jt_p\omega/T_Q}, \quad |\omega| \leq \pi \tag{12}$$

for $p = 0, 1, \ldots N-1$.

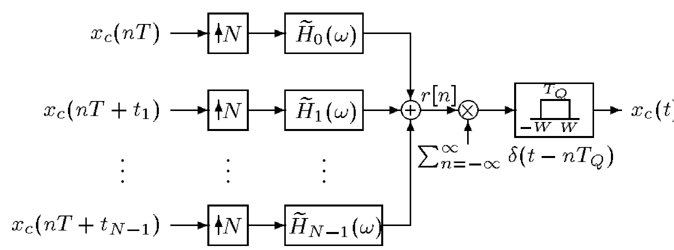

Figure 6: Reconstruction from recurrent nonuniform samples using a DTFB.

As with the CTFB of Fig. 5, the overall output of Fig. 6 is the original CT signal $x_c(t)$. Furthermore, since $x_c(t)$ is reconstructed through low-pass filtering of a uniformly spaced impulse train, the impulse train values $r[n]$ must correspond to uniformly spaced samples of $x_c(t)$ at the Nyquist rate. Thus, the DTFB of Fig. 6 effectively interpolates the recurrent nonuniform samples to uniform Nyquist samples. The DTFB of Fig. 6 can be used to reconstruct the CT signal from its recurrent nonuniform samples. As with the CTFB, the magnitude responses of the DT filters are piecewise constant, which allows for further efficiency in the implementation.

5. GENERATING NEW SAMPLING THEOREMS

The Interpolation Identity can be used to convert the reconstruction (synthesis) part of a CTFB to an equivalent DTFB followed by impulse modulation and low-pass filtering. Similarly, we can convert the sampling (analysis) part of the filter bank using the equivalence of Fig. 7(a) and 7(b) for any $x_c(t)$, $h(t)$ bandlimited to $W = \pi/T_Q$, where

$$\widetilde{H}(\omega) = H\left(\frac{\omega}{T_Q}\right), \quad |\omega| \leq \pi. \tag{13}$$

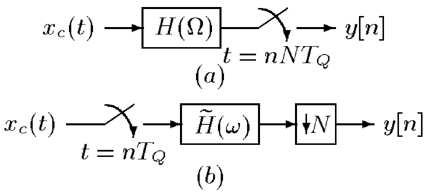

Figure 7: Sampling equivalence.

The equivalence of Fig. 7 follows in a straightforward way by noting that sampling a CT signal at one-N'th of the Nyquist rate can be realized by sampling the signal at the Nyquist rate followed by decimation by a factor of N. We can then apply the known result ([4]) regarding DT processing of a CT signal to replace the CT filter by a DT filter with frequency response given by (13), operating on Nyquist rate samples of the CT signal.

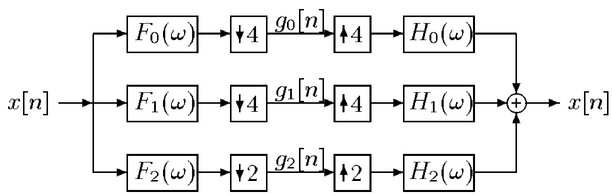

Figure 8: Perfect reconstruction DTFB.

The Interpolation Identity of Fig. 2 together with the equivalence of Fig. 7 enables us to convert any CTFB to an equivalent DTFB preceded by Nyquist rate sampling and followed by low-pass filtering, and vice versa. Thus, any perfect reconstruction (PR) filter bank (i.e. a DT analysis-synthesis filter bank for which the input and output are equal) can be converted to a CTFB, which can then be interpreted in terms of sampling and reconstruction.

As an example consider the PR filter bank of Fig . 8. The theory of PR filter banks is well established (see e.g. [6]) and closed form solutions for the synthesis filters $H_p(\omega)$ given the analysis filters $F_p(\omega)$ are known. We can convert the analysis part of the filter bank to a sampling strategy by applying the equivalence of Fig . 7. This results in the sampling strategy depicted in Fig. 9(a), where the signal $x_c(t)$ is filtered by three CT filters with frequency responses $F_p(\Omega T_Q)$, $p = 0, 1, 2$, and the outputs are sampled at the corresponding rates. The reconstruction is obtained by applying the Interpolation Identity of Fig. 2 to the synthesis part of the filter bank followed by impulse modulation and low-pass filtering, resulting in the reconstruction depicted in Fig. 9(b).

The sampling procedure of Fig. 9(a) together with the reconstruction of Fig. 9(b) constitute a generalization to Papoulis' well known generalized sampling expansion ([5]). Papoulis showed that a bandlimited signal $x_c(t)$ is uniquely determined by the samples $g_k(nT)$ of the responses $g_k(t)$ of M LTI systems with input $x_c(t)$, sampled at one-M'th of the Nyquist rate. By converting a PR filter bank with unequal decimation factors to a sampling and reconstruction scheme, we allow for different sampling rates of the filters outputs, thus generalizing Papoulis' theorem.

Papoulis does not derive necessary and sufficient conditions on the filters such that the signal can be reconstructed from the generalized samples. However, such conditions can be derived by using (13) to convert the CT filters to DT filters comprising a DT filter bank. Given the analysis filters of a DTFB, we can determine if synthesis filters ensuring PR exist ([6]), i.e. if the signal can be reconstructed from samples of the filters outputs.

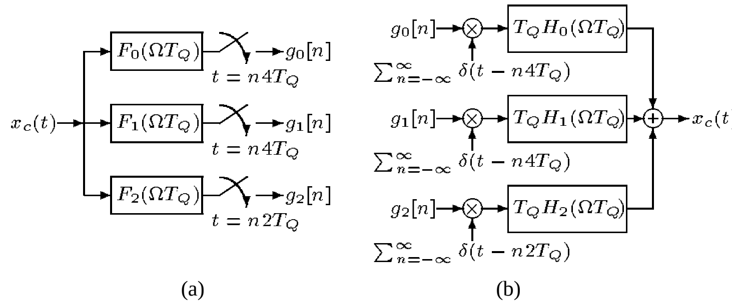

Figure 9: (a) Sampling procedure. (b) Reconstruction using a CTFB.

6. CONCLUSION

This paper introduces a filter bank interpretation of various sampling methods, thereby allowing for efficient implementation of the reconstruction from generalized samples as well as from recurrent nonuniform samples. The block diagram equivalences formulated in this paper are general in the sense that they can be used to convert arbitrary CTFBs to equivalent DTFBs and vise versa. Presenting the reconstruction from generalized samples in terms of CT filters and applying the Interpolation Identity leads to efficient implementations that inherently interpolate the uniform Nyquist samples of the signal. Furthermore, the equivalences provide additional insight into the sampling and reconstruction process, thus leading to a whole new class of sampling theorems.

7. REFERENCES

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