

DISCRETE-TIME RANDOMIZED SAMPLING

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ABSTRACT

This paper explores the use of randomized sampling in implementing convolution in discrete-time with application to the areas of approximate filtering, low-power filter design, and hardware failure modeling. Three distinct randomized sampling methods are presented and additive error models as well as second-order error statistics are derived for these for both white and semi-correlated sampling processes. Discrete-time randomized sampling (RS) is then considered as a filter approximation method and conditions are derived under which RS-based approximations to the Wiener filter lead to a smaller mean-square estimation error than the best constrained LTI approximation. The tradeoff between power savings and output quality is also investigated for low-power applications. In addition, the RS framework is used to model a class of random hardware failures and algorithms are presented to improve the output SNR.

1. INTRODUCTION

Signal processing algorithms are designed to satisfy a variety of constraints depending on the application and the resources available [1]. For example, when the amount of computation or the processing time is an issue, algorithms should be optimized for computational or time efficiency while low-power algorithms are designed for use in environments where power is limited such as battery-powered mobile devices. In this paper, we present a framework based on randomly sampling signals or the impulse response of LTI filters which allows the modification of pre-existing algorithms to meet certain complexity, robustness or power constraints. Since this approach does not require re-computation of the filter coefficients, it is not computationally intensive and therefore is particularly attractive in applications where real-time adjustments to changing resources is needed.

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Randomized sampling is defined within the broader randomized signal processing framework used when the signal conversion or processing procedures are performed stochastically. When the sampling is performed at random time-intervals, it is described as randomized. Most of the randomized sampling literature starts from a continuous-time signal and thus consists of a randomization of the sampling process [2]. In this paper, the randomized sampling is carried out exclusively in discrete-time and can therefore be thought of as randomized down-sampling.

In the next sections, we start by defining discrete-time randomized sampling and deriving models and properties for the resulting errors. This framework is then applied to the problem of approximating Wiener filters and conditions are derived under which randomized sampling methods perform better, in a mean-square sense, than more traditional approximation methods. We also explore randomized sampling in a low-power digital filtering context. Power savings are evaluated and the trade-off between power and system performance is analyzed. We then consider hardware failure models in which the distortion introduced by the faulty system can be represented by the randomized sampling algorithms, and present simple techniques that guarantee a desired performance level under a given probability of failure.

2. DEFINITIONS

Given an LTI system with a wide-sense-stationary stochastic input $x[n]$ and deterministic impulse response $h[n]$, we define as *discrete-time randomized sampling* the process of randomly setting some of the time samples of the input or the filter impulse response to zero and denote it *RS* with the understanding that the sampling is performed exclusively in discrete-time. There are three forms of RS: randomized sampling of the input, randomized sampling of the filter impulse response, and iterative randomized sampling of the filter impulse response.

- **Randomized sampling of the input (RSI).**
Let $r_s[n]$ be a wide-sense-stationary stochastic

process with mean m_r , independent of $x[n]$, and that can only take on values zero or one. RSI refers to the process of multiplying the input $x[n]$ by $r_s[n]$ and then filtering the result with $h[n]$, i.e. the resulting output, $y_1[n]$, can be written as:

$$y_1[n] = \sum_{k=-\infty}^{+\infty} r_s[k]x[k]h[n-k]. \quad (1)$$

- **Randomized sampling of the filter impulse response (RSF).**

If the process $r_s[n]$ is multiplied by the filter impulse response, $h[n]$, then the resulting scheme is referred to as RSF. In this case, the output is given by:

$$y_2[n] = \sum_{k=-\infty}^{+\infty} r_s[k]h[k]x[n-k]. \quad (2)$$

- **Iterative randomized sampling of the filter impulse response (IRSF).**

A more elaborate sampling technique consists of randomly sampling the filter impulse response at each iteration of the convolution sum. The output can then be written as:

$$y_3[n] = \sum_{k=-\infty}^{+\infty} r_s[n;k]h[k]x[n-k]. \quad (3)$$

We restrict $r_s[n;k]$ to be a wide-sense-stationary two-dimensional stochastic process with mean m_r , independent of $x[n]$, and that can only take on values zero or one.

3. LINEAR ERROR MODELS

Since the desired system has output $y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$, any of the three RS techniques will result in some error. In each case, it can be shown [3] that the RS output can be expressed as $y[n]$ scaled by m_r , added to an error term $e[n]$ which is uncorrelated with $y[n]$. In addition, if the sampling is white, the error sequence $e[n]$ is white only for IRSF. Table 1 gives the second-order error statistics for the three sampling methods using white sampling sequences.

A generalization of the IRSF white sampling approach which we refer to as semi-correlated sampling consists of using a sampling process that is white in one dimension and correlated in the other. This results in two different sampling strategies. We first consider the case in which the processes used to sample the impulse response in order to compute successive output samples are uncorrelated with each other, i.e. $r_s[n;k]$ is white in the n -dimension

Method	$C_{r_s r_s}[m]$ and $C_{r_s r_s}[m;l]$	$C_{ee}[m]$
RSI	$\sigma^2 \delta[m]$	$\sigma^2 C_{xx}[0]C_{hh}[m]$
RSF	$\sigma^2 \delta[m]$	$\sigma^2 C_{hh}[0]C_{xx}[m]$
IRSF	$\sigma^2 \delta[m]\delta[l]$	$\sigma^2 C_{xx}[0]C_{hh}[0]\delta[m]$

Table 1: Second-order error statistics for the different sampling methods using white sampling functions. $C_{r_s r_s}[m]$ and $C_{r_s r_s}[m;l]$ are the sampling processes covariance functions, and $\sigma^2 = m_r(1 - m_r)$.

but arbitrary in the k -dimension. The covariance function of $r_s[n;k]$ is thus given by $C_{r_s r_s}[m;l] = m_r(1 - m_r)\delta[m]C_{gg}[l]$ where m_r is the mean of the process and $C_{gg}[l]$ is a one-dimensional covariance function that is not restricted to be white and describes the second-order statistics of $r_s[n;k]$ in the k -dimension. It can be shown that the error resulting from such sampling is still white with variance given by the first entry in Table 2. In addition, if $C_{gg}[l]$ represents a first-order process, i.e. $C_{gg}[l] = \delta[l] - \alpha\delta[l-1] - \alpha\delta[l+1]$ where $-1 < \alpha < 1$, then the covariance of the error is given by the second entry in Table 2. Alternatively we can choose $r_s[n;k]$ to be white in the k -dimension i.e. white at each convolution step but correlated in the n -dimension. In this case, the two-dimensional covariance function of the sampling process can be written as $C_{r_s r_s}[m;l] = m_r(1 - m_r)C_{vv}[m]\delta[l]$, where m_r is the mean of the process and $C_{vv}[m]$ is a one-dimensional covariance function that is not restricted to be white and describes the second-order statistics of $r_s[n;k]$ in the n -dimension. The resulting error has a covariance function proportional to the product of the covariance of the input and $C_{vv}[m]$ as shown in the third entry in Table 2. If $C_{vv}[m]$ is a first-order process, i.e. $C_{vv}[m] = \delta[m] - \alpha\delta[m-1] - \alpha\delta[m+1]$ where again $-1 < \alpha < 1$, then the covariance of the error is given by the last entry in Table 2.

4. ERROR ANALYSIS

As indicated in Table 1, in the white sampling case, the error sequences resulting from all three sampling schemes have the same total power: $m_r(1 - m_r)C_{xx}[0]C_{hh}[0]$ and therefore they all lead to the same full-band output signal-to-noise ratio (SNR). However, the errors have different spectral densities and therefore lead to different in-band SNR, i.e. for bandlimited outputs, the ratio of the power in the output signal to the noise power within the signal band is different across sampling methods. In fact, IRSF always leads to a higher in-band signal-to-noise ratio than RSI if the filter is bandlimited. If, on the other hand, the input has a bandlimited power spectrum, then IRSF always leads to a higher

$C_{r_s r_s}[m; l]$	$C_{ee}[m]$
$\sigma^2 \delta[m] C_{gg}[l]$ if $C_{gg}[l] = \delta[l] - \alpha \delta[l-1] - \alpha \delta[l+1]$	$\sigma^2 \sum_k \sum_l h[k] h[l] C_{gg}[l-k] C_{xx}[l-k] \delta[m]$ $\sigma^2 (C_{hh}[0] C_{xx}[0] - 2\alpha C_{hh}[1] C_{xx}[1]) \delta[m]$
$\sigma^2 C_{vv}[m] \delta[l]$ if $C_{vv}[m] = \delta[m] - \alpha \delta[m-1] - \alpha \delta[m+1]$	$\sigma^2 C_{hh}[0] C_{vv}[m] C_{xx}[m]$ $\sigma^2 C_{hh}[0] C_{xx}[0] (\delta[m] - \alpha \frac{C_{xx}[1]}{C_{xx}[0]} \delta[m-1] - \alpha \frac{C_{xx}[1]}{C_{xx}[0]} \delta[m+1])$

Table 2: Second-order error statistics for IRSF using semi-white sampling processes. $C_{r_s r_s}[m; l]$ is the sampling process covariance function, $C_{gg}[l]$ and $C_{vv}[m]$ are covariance functions that describe the correlation of the two-dimensional sampling process, $r_s[n; k]$, in the k - and n -dimensions respectively, and $\sigma^2 = m_r(1 - m_r)$.

in-band SNR than RSF. In addition, it follows that in the case where the filter is bandlimited to ω_0 , if the ratio of input power within the filter band to total power in the input is greater than $\frac{\omega_0}{\pi}$, then IRSF always leads to a higher in-band SNR than RSF, and thus in this case, IRSF leads to the highest in-band SNR. Similar reasoning can be applied to the case where the input power is bandlimited to ω_0 . There are therefore clear advantages to using IRSF over other sampling methods.

The results in Table 2 further suggest that improved error behavior can be obtained if appropriate correlation is introduced in the sampling process. For example, if $\alpha = \text{sign}\{C_{hh}[1]C_{xx}[1]\}$ in the first semi-correlated sampling example with error covariance given by the second entry in Table 2, then the resulting error has a lower variance than the error resulting from white sampling, therefore leading to improved SNR. Similarly, while the total power in the error resulting from the second semi-correlated sampling example with error covariance given by the last entry in Table 2 is the same as the white sampling case, if $\alpha = \text{sign}\{C_{xx}[1]\}$, the error power is highest at high frequencies whereas if $\alpha = -\text{sign}\{C_{xx}[1]\}$, the error power is highest at low frequencies. This spectral shaping is reminiscent of the error behavior in a sigma-delta A/D converter and can be exploited to shape the IRSF error and therefore increase the in-band SNR.

5. RS-BASED APPROXIMATIONS TO WIENER FILTERS

One direct application of RS consists of reducing the complexity of a filter by lowering the amount of computation (number of non-zero multiplications) needed per output sample therefore trading-off quality for computation. In this section, we consider the application of IRSF to approximate Wiener filters. We also compare the performance of IRSF-based approximations to traditional methods corresponding to approximating the original LTI filter of length P by a shorter LTI filter of length M , where $M < P$. Specifically, the objective is to obtain an approximation to the best linear mean-square estimator

(the Wiener filter) of the signal $y[n]$ based on the received data, $x[n]$. Using the mean-square estimation error as a performance metric, we wish to define conditions under which IRSF-based approximations perform better than traditional approximations, i.e. lead to a smaller increase in mean-square estimation error. Block diagrams for the different approximation techniques are shown in Figure 1. Note that for the case of IRSF, the output is rescaled in order to express the output of the system as the original output added to an error term.

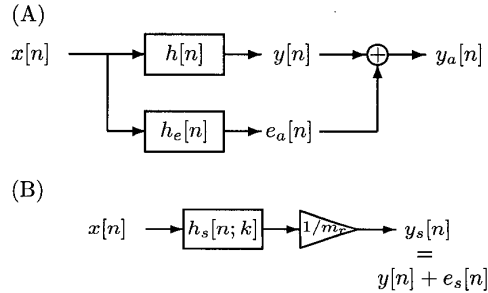


Figure 1: Block diagram of the LTI (A) and IRSF (B) approximations of $h[n]$. $y[n]$ denotes the original output of the Wiener filter, $e_a[n]$ and $e_s[n]$ are the approximation errors for the LTI and the IRSF approximation methods respectively, $h_s[n; k]$ is the sampled version of $h[n]$ using IRSF, and m_r is the mean of the sampling process.

In the following, we refer to the difference between the output of the exact Wiener filter and the output of the approximated filter as the *approximation error* whereas the *estimation error* refers to the difference between the desired signal and the output of the estimator. It can be shown [3] that the increase in mean-square estimation error is equal to the variance of the approximation error for both the traditional and the IRSF methods. As a result, the variance of the approximation errors is a sufficient measure for this problem.

5.1 Zero-Mean, Unit-Variance, White Process, $x[n]$

If the data is white, truncating the Wiener filter impulse response results in the LTI filter with smallest mean-square approximation error for a given filter length M . In this case, IRSF using white sampling outperforms the truncation method and therefore *any* approximation technique leading to an FIR filter of length M if and only if:

$$0 < \frac{1 - m_r}{2m_r - 1} < \frac{\sum_k (h[k] - h_w[k])^2}{\sum_k h_w^2[k]} \quad (4)$$

where $h_w[n]$ is the truncated version of $h[n]$ and m_r is the mean of the sampling process and is equal to the ratio of the length of the original filter, $h[n]$, to the length of the truncated filter, $h_w[n]$. As a result, if the ratio of the total energy in the portion of the impulse response not included in the truncated filter to the total energy in the truncated impulse response is greater than $\frac{1-m_r}{2m_r-1}$, then IRSF leads to an estimation error with lower variance than the filter approximation using truncation.

5.2 Non-White Process, $x[n]$

The truncated Wiener filter is the best length- M approximation to the linear minimum mean-square estimator only if the received data, $x[n]$, is white. If $x[n]$ is not white, then the optimal linear mean-square estimator needs to be computed every time M changes. This may not be desirable or even feasible in cases where resources are limited or if algorithm simplicity is of primary concern. In these cases, a straight-forward approach would be to truncate the Wiener filter or iteratively randomly sample it if the conditions in equation (4) are satisfied. Alternatively, the estimator could be decomposed into two systems with different resource allocations: a whitening filter implemented accurately, and a constrained Wiener filter (with limited resources) operating on the whitened data. In this case, the problem is similar to the one discussed in the previous section and the IRSF method leads to a better mean-square estimate if the conditions of equation (4) are satisfied.

6. OTHER APPLICATIONS

6.1 Low-Power Digital Filtering

Since randomized sampling leads to a reduction in non-zero terms in the convolution sum, less power can be consumed to compute the output therefore leading to a reduction of the switching activity if one checks for multiply-by-zero operations. Output quality can thus be traded for power savings. Figure 2 illustrates a low-power implementation for

discrete-time randomized sampling. At each time point or clock cycle, a subset of the outputs of the multipliers is picked by a collection of multiplexers (abstracted by MUX in the figure), the remaining multiplications are not performed therefore reducing computation. The chosen subset is then used as an input to an accumulator which performs the final addition to generate the output.

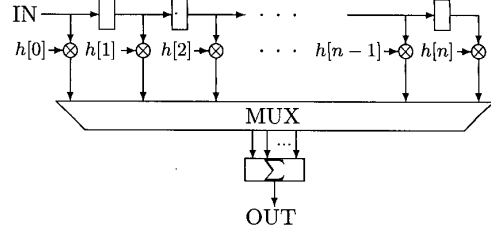


Figure 2: FIR filter structure for low-power filtering using discrete-time randomized sampling. The rectangles represent delay registers and MUX is used as an abstraction to a collection of multiplexers.

Since the computational load is decreased, throughput can be maintained while reducing the supply voltage resulting in additional power savings. Relative power savings as a function of the mean of the sampling process is shown in Figure 3 while Figure 4 quantifies the trade-off between power consumption and output quality using SNR as a quality measure. Specifically, SNR (in dB) is given by $10 \log_{10}(\frac{m_r}{1-m_r}) + 10 \log_{10}(\frac{C_{yy}[0]}{C_{xx}[0]C_{hh}[0]})$, where the mean of the sampling process only appears in the first term which we refer to as gain in SNR and use as a performance measure in Figure 4. Note that if none of the filter coefficients are set to zero, i.e. if $m_r = 1$, the gain in SNR is infinite as expected.

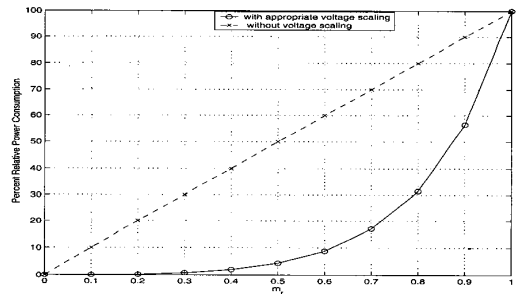


Figure 3: Relative percent power consumption as a function of the sampling process mean m_r . [Chandrakasan *et al.*[4] derive a similar relationship where the relative power consumption is plotted as a function of the normalized workload.]

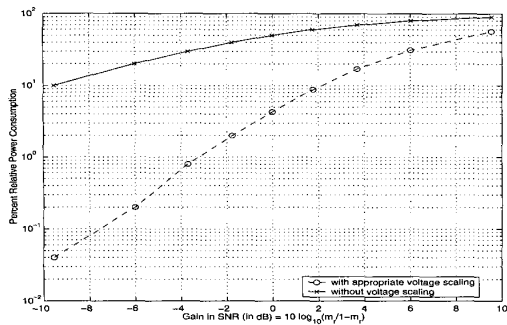


Figure 4: Relative percent power consumption as a function of the gain in SNR at the output on a semi-log scale.

6.2 Modeling Hardware Failures

Randomized sampling can also be used to model certain types of hardware failures. Specifically, in the case of direct and transposed direct form implementations of FIR filters and for a given time point n and a given non-zero filter coefficient $h[i]$, the following three types of failures generate the same error and are therefore equivalent from a distortion point of view:

1. The value of $h[i]$ is read as a zero.
2. The output of the multiplier associated with $h[i]$ is incorrectly set to zero.
3. An addition is skipped by not using the value resulting from the multiplier operation associated with $h[i]$.

If for each coefficient $h[i]$ the probability of an error occurring resulting from one of the above modes of failures is $(1 - m_r)$ and is independent of the occurrence of other similar errors associated with other coefficients, then the resulting faulty process is equivalent to white iterative randomized sampling of the FIR impulse response. In particular, under these circumstances and after proper scaling, the distortion introduced consists of additive zero-mean white noise at the output with variance $\frac{1-m_r}{m_r} C_{xx}[0] C_{hh}[0]$.

Simple techniques can be used to improve the output SNR. For example, the input signal can be up-sampled by a factor of N and filtered at a higher rate and the resulting output low-pass filtered using more reliable hardware as shown in Figure 5. Alternatively, N parallel filters could be used followed by averaging the outputs of the filters. In both cases, the SNR (in dB) is given by $10 \log_{10}(\frac{Nm_r}{1-m_r}) + 10 \log_{10}(\frac{C_{yy}[0]}{C_{xx}[0] C_{hh}[0]})$, where the effects of N and m_r on the SNR only appear in the first term which we denote as gain in SNR. As an example, up-sampling by a factor of 10 (or using 10 parallel filters) will be needed to maintain a gain in SNR of 20dB under a 1% probability of failure.

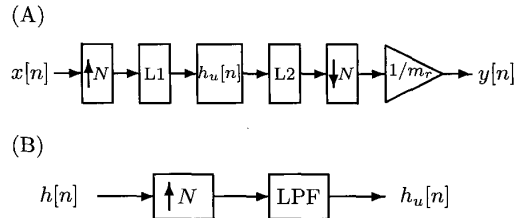


Figure 5: (A) Up-sampled implementation for hardware failure applications. L1 and L2 are low-pass filters with cutoff $\frac{\pi}{N}$ and gain N and 1 respectively. It is assumed that only $h_u[n]$ is implemented on the faulty hardware. (B) Definition of $h_u[n]$. LPF is a low-pass filter with cutoff $\frac{\pi}{N}$ and gain N .

7. CONCLUSION

This paper explored a randomized sampling approach to implementing convolution in discrete-time and its applications to three areas: approximate filtering, low-power filter design, and modeling hardware failure. It was shown that correlated sampling can lead to better error performance than white sampling and conditions were defined where an iterative discrete-time randomized sampling approach leads to a better approximation to the Wiener filter, in terms of mean-square estimation error or approximation error, than any LTI approximation technique. Significant power savings for low-power applications were also suggested through lower switching activity and supply voltage scaling. It should be further noted that because of its low complexity, this approach is particularly attractive for applications where real-time adjustments are needed to adapt to changing resources. Finally, we showed that specific hardware failures could be modeled using this framework and presented simple algorithms to improve the quality of the output under a given probability of failure.

8. REFERENCES

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