

DETECTION AND ESTIMATION OF SOLITON SIGNALS

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ABSTRACT

Soliton solutions to nonlinear wave equations have been recently proposed as signaling waveforms in a variety of communication contexts. One such system modulates the relative positions or amplitudes of multiple solitons generated by a nonlinear ladder circuit. At the receiver, there are inherent difficulties in the problems of parameter estimation and detection of soliton signals due to the nonlinear coupling imposed by the soliton dynamics. In this paper, we demonstrate that the ladder circuit can act as a tuned receiver for the component solitons, naturally decoupling them so that the detection and estimation problems can be solved with standard techniques. We develop robust and asymptotically efficient algorithms for maximum likelihood parameter estimation and present a technique for generalized likelihood ratio test detection.

1. INTRODUCTION

Solitons arise in a variety of natural phenomena including water waves, anharmonic crystal lattice vibrations, and pressure waves in liquid-gas bubble mixtures [1]. They are also present in a number of man-made media such as superconducting transmission lines and optical transmission in nonlinear fibers [2]. Several intriguing experiments have also been conducted using nonlinear circuits to generate soliton carrier signals for modulation of information [3]–[8].

Solitons are stable, localized solutions to nonlinear wave equations that propagate with constant shape and velocity. The collision of two solitons reveals a particle-like behavior as each emerges from the collision virtually unchanged. In fact, solitons can be viewed as the normal modes of certain nonlinear systems as these solutions satisfy a nonlinear form of superposition. Although these systems are nonlinear, they are exactly solvable through a technique known as “inverse scattering,” which decouples the nonlinear modes and can be viewed as an analog of the Fourier transform [9].

The prevalence of systems exhibiting soliton behavior, both natural and man-made, indicates a wide range of contexts in which one is interested in solving basic signal processing problems with soliton signals. In particular, the

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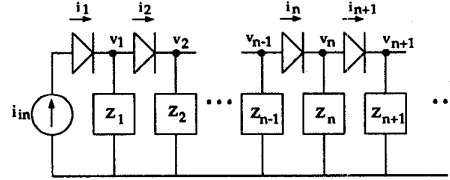


Figure 1. Diode ladder implementation of the Toda Lattice.

problems of parameter estimation and detection arise naturally in the analysis of modulation system performance.

In this paper, the problems of parameter estimation and detection of soliton signals in the presence of additive corruption are considered. We will work in the framework of the Toda lattice circuit shown in Fig. 1, where $z_n = 1/s^2$ is a “double capacitor.” In [4], it is shown that the diode ladder circuit satisfies the Toda lattice equations [10],

$$\frac{d^2}{dt^2} \ln(1 + i_n) = (i_{n-1} - 2i_n + i_{n+1}), \quad (1)$$

where i_n is the current through the n th diode, $i_0 = i_{in}$, and for simplicity, the parameters of all circuit elements have been normalized to unity. A single soliton will propagate along the lattice when the input is

$$i_{in}(t) = \beta^2 \operatorname{sech}^2(\beta(t - \delta)), \quad (2)$$

which results in (1) having the solution

$$i_n(t) = \beta^2 \operatorname{sech}^2(\beta(t - \delta) - pn), \quad (3)$$

with the dispersion relation $\beta = \sinh(p)$. Note that the solitons propagate with a velocity, $c = \beta/p$, that is dependent on the scale parameter, β . This implies that tall, narrow solitons travel faster than short, wide solitons.

For the remainder of this paper, a simplified channel model will be assumed, in which a received waveform, $r(t) = s(t) + n(t)$ comprises the transmitted signal, $s(t)$, in stationary white Gaussian noise, $n(t)$, with power N_0 . When $r(t)$ is the input to the Toda lattice circuit, i.e. $i_0(t) = r(t)$, the signal $i_n(t) = s_n(t) + n_n(t)$ may be defined as the n th diode current comprising soliton and noise components $s_n(t)$ and $n_n(t)$ respectively. It can be shown that at high signal-to-noise ratios, the soliton component is relatively unaffected by the noise while the noise component is Gaussian and low pass [6].

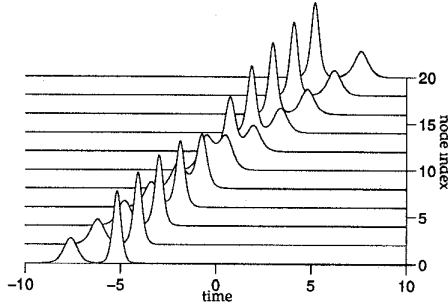


Figure 2. A two-soliton solution to the Toda lattice.

We will focus our attention on the two-soliton signal,

$$s(t) = \frac{\beta_1^2 \operatorname{sech}^2(\eta_1) + \beta_2^2 \operatorname{sech}^2(\eta_2) + A \operatorname{sech}^2(\eta_1) \operatorname{sech}^2(\eta_2)}{(\cosh(\phi/2) + \sinh(\phi/2) \tanh(\eta_1) \tanh(\eta_2))^2},$$

$$A = \sinh(\phi/2) \left((\beta_1^2 + \beta_2^2) \sinh(\phi/2) + 2\beta_1\beta_2 \cosh(\phi/2) \right),$$

$$\phi = \ln \left(\frac{\sinh((p_1 - p_2)/2)}{\sinh((p_1 + p_2)/2)} \right), \quad (4)$$

where $\beta_i = \sinh(p_i)$ and $\eta_i = \beta_i(t - \delta_i)$. In Fig. 2, the diode currents, $i_n(t)$, are plotted as a function of time and node index for $\beta_1 = \sinh(2)$ and $\beta_2 = \sinh(1.5)$. Note that as the tall soliton passes through the short soliton, the amplitude of their nonlinear superposition is actually smaller than the sum of their individual amplitudes. It can be shown [6] that when the two solitons precisely overlap, $\delta_1 = \delta_2$, both peak power and average power are minimized. The nonlinear interaction of the component solitons can also enhance the parameter estimation error performance [6]. Both of these properties make the superimposed solitons particularly attractive for a variety of communications contexts.

2. ESTIMATION ALGORITHMS

2.1. General Approach

In this section we will present and analyze the performance of several algorithms for estimating the parameters of soliton signals. Consider the problem of estimating the position, δ , of a single soliton,

$$s(t; \delta) = \beta^2 \operatorname{sech}^2(\beta(t - \delta)), \quad (5)$$

with the parameter β known. For observations in stationary white Gaussian noise, the maximum likelihood (ML) estimate is given by the value of the parameter θ that maximizes the correlation,

$$\hat{\delta} = \arg \max_{\theta} \int_{t_i}^{t_f} r(t) s(t; \theta) dt. \quad (6)$$

It is well-known that an efficient way to perform the correlation (6) with all of the replica signals $s(t; \theta)$ over the range $\delta_{\min} < \theta < \delta_{\max}$, is a matched filter.

When the signal $r(t)$ contains a multi-soliton signal, $s(t; \underline{\beta}, \underline{\delta})$ and we wish to estimate the parameter vector $\underline{\delta}$, the estimation problem becomes more involved. If the component solitons are well separated in time, then the maximum likelihood estimator for the positions of each of the

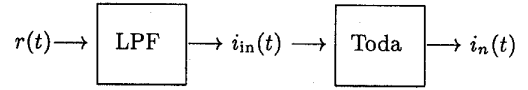


Figure 3. Toda lattice receiver model.

component solitons would again involve a matched filter processor followed by a peak-detector for each soliton.

If the component solitons are not well-separated and are therefore nonlinearly combined, such as on the 10th node in Fig. 2, a better approach is needed. The estimation of δ_1 and δ_2 should not be performed independently. The maximum likelihood processor would correspond to a minimization of the difference between the observed signal $r(t)$ and a replica signal $s(t; \underline{\beta}, \underline{\delta})$ over the parameter space,

$$\hat{\underline{\delta}} = \arg \min_{\underline{\delta}} \int_{t_i}^{t_f} (r(t) - s(t; \underline{\beta}, \underline{\delta}))^2 dt. \quad (7)$$

The estimation problems can be conveniently decoupled by preprocessing the signal $r(t)$ with the Toda lattice. By setting $i_{\text{in}}(t) = r(t)$, as the signal $i_n(t)$ propagates through the lattice, the component solitons will naturally separate. Since the noise component remains Gaussian and low pass [5] the ML estimator, $\hat{\delta}_{\text{ML}}(i_n)$, reduces to a set of matched filters, one for each of the component solitons. The invertibility of the lattice equations via inverse scattering guarantees that the ML estimate $\hat{\delta}_{\text{ML}}(r(t))$ will be the same as the estimate $\hat{\delta}_{\text{ML}}(i_n(t))$. This is based on the well-known invariance property of the ML estimator.

For the purposes of our simulations, we assume that the receiver comprises a low pass filter followed by a Toda lattice circuit as shown in Fig. 3 and that the bandwidth, $2\pi/\Delta$, of the low pass filter in Fig. 3 is wide enough to pass the soliton component of $r(t)$ completely. The input $i_{\text{in}}(t)$ to the Toda lattice circuit then comprises the soliton signal in low pass Gaussian noise. Simulations of the algorithms were performed using a Runge-Kutta integration routine with a fixed step size, Δ . To model the effects of the noise, an i.i.d. Gaussian random sequence, $w(k\Delta) \sim N(0, \sigma_w^2)$, was added to the samples of the input sequence $i_{\text{in}}(k\Delta)$ resulting in an effective white noise power of $N_0 = \Delta \sigma_w^2$.

2.2. Position estimation

If the component solitons separate by the N th node, the signal will appear to be a linear superposition of two solitons,

$$s_N(t) \approx \beta_1^2 \operatorname{sech}^2(\beta_1(t - \delta_1) - p_1 N - \phi/2) \quad (8)$$

$$+ \beta_2^2 \operatorname{sech}^2(\beta_2(t - \delta_2) - p_2 N + \phi/2), \quad (9)$$

where $\phi/2$ is a shift incurred due to the nonlinear interaction. Matched filters can now be used to detect the arrival of each soliton at the N th node. We formulate the estimate

$$\hat{\delta}_1 = \left(t_{N,1}^a - \frac{p_1 N + \phi/2}{\beta_1} \right), \quad \hat{\delta}_2 = \left(t_{N,2}^a - \frac{p_2 N - \phi/2}{\beta_2} \right), \quad (10)$$

where $t_{N,i}^a$ is the time of arrival of the i th soliton at node N . The performance of this algorithm for a two-soliton signal with $\underline{\beta} = [\sinh(2), \sinh(1.5)]$ is shown in Fig. 4. As would be expected, the position estimates of the tall soliton

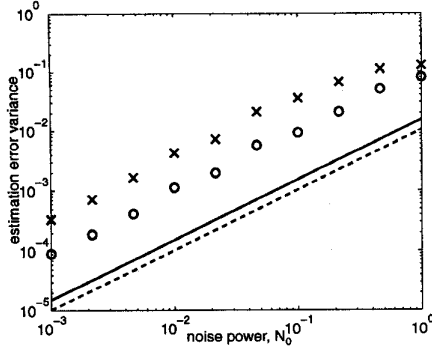


Figure 4. The CRB for δ_1 and δ_2 are shown with solid and dashed lines, while the estimation error results of 100 Monte-Carlo trials are indicated with 'o' and 'x' marks, respectively.

are superior to those of the short soliton. However, since the solitons in $r(t)$ are overlapping, the Cramér-Rao bound (CRB) for the position of the small soliton is, in fact, less than that of the larger[6]. Note that although the error variance of each estimate appears to be a constant multiple of the CRB, the estimation error variance approaches the CRB in an absolute sense as $N_0 \rightarrow 0$, indicating that the estimates are asymptotically efficient.

A sufficient statistic for the estimation of β based on observations of a single soliton in Gaussian noise does not exist due to the highly nonlinear manner in which the parameter appears both in the time and the amplitude scales of the signal [6]. Although an ML solution might be found by numerical maximization of the likelihood function, such a solution would be computationally intensive. However, through the framework of inverse scattering, we can obtain estimates that appear empirically to be both unbiased and asymptotically efficient.

2.3. Inverse Scattering Based Estimation

Inverse scattering theory for the Toda lattice demonstrates that the eigenvalues of the matrix,

$$L(t) = \begin{bmatrix} & & & & & \\ & & & & & \\ & & a_{n-2} & & & \\ a_{n-2} & b_{n-1} & a_{n-1} & & & \\ & a_{n-1} & b_n & a_n & & \\ & & & & \ddots & \\ & & & & & \end{bmatrix}, \quad (11)$$

are time-invariant, where $a_n = \frac{1}{2}e^{(v_n - v_{n+1})/2}$, $b_n = \dot{v}_n/2$, and $v_n(t)$ are the node voltages for any solution to the Toda lattice [10]. Further, the eigenvalues of $L(t)$ for which $|\lambda_i| > 1$ correspond to soliton solutions, with $\beta_i = \sqrt{\lambda_i^2 - 1}$.

A natural algorithm for jointly estimating the parameters β_i for a single or multi-soliton solution arises. By processing the received signal $r(t)$ with the Toda lattice, the sequences a_n and b_n may be obtained from measurements of the circuit. The parameter estimation algorithm now amounts to the estimation of the eigenvalues of $L(t)$.

The algorithm performance for the joint estimation of the parameters $\beta_1 = \sinh(2)$ and $\beta_2 = \sinh(1.5)$ of a two-soliton signal is shown in Fig. 5.

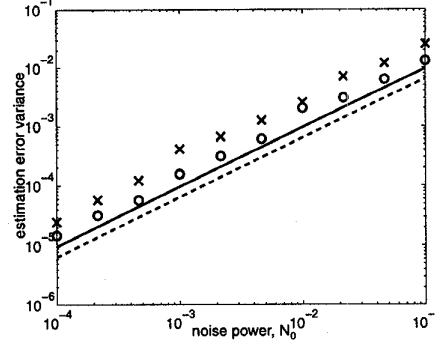


Figure 5. The CRBs for β_1 and β_2 are shown with solid and dashed lines, while the estimation error results of 100 Monte-Carlo trials are indicated with 'o' and 'x' marks, respectively.

3. DETECTION OF SOLITON SIGNALS

The problem of detecting a single soliton or multiple non-overlapping solitons falls within the theory of classical detection. When the signal $r(t)$ contains a multi-soliton signal in which the component solitons are not resolved, the classical approach to detection becomes more complex. If the relative positions of the component solitons are known *a priori*, then the detection problem reduces to deciding which among several possible signals is present. For a two-soliton signal in stationary white Gaussian noise, $n(t)$, we have

$$\begin{aligned} H_0 &: r(t) = n(t), \\ H_1 &: r(t) = s_1(t) + n(t), \\ H_2 &: r(t) = s_2(t) + n(t), \\ H_{12} &: r(t) = s_{12}(t) + n(t), \end{aligned}$$

where $s_1(t)$, $s_2(t)$, $s_{12}(t)$ are soliton 1, soliton 2 and the two-soliton signals respectively. If the relative positions of the solitons are unknown, then the signal $s_{12}(t)$ will vary significantly as a function of the relative separation, $\delta_1 - \delta_2$ (see Fig. 2). The general problem of detection with an unknown parameter, $\underline{\delta}$, can be handled in a number of ways. If the parameter can be modeled as random and the distribution for the parameter is known, $p_{\underline{\delta}}(\underline{\delta})$, along with the distributions $p_{r|\underline{\delta}, H}(R|\underline{\delta}, H_i)$ for each hypothesis, then the Bayes or Neyman-Pearson criteria can be used. Unfortunately, even when the distribution for the parameter $\underline{\delta}$ is known, the likelihood ratios cannot be found in closed form.

Another approach that is commonly used is to assume that the value of the unknown parameter $\underline{\delta}$ is equal to its ML estimate. Such techniques are called "generalized likelihood ratio tests" (GLRT) and perform well in practice when the likelihood function has a sharp peak near its maximizing value, $\hat{\underline{\delta}}_{ML}$. If we employ a GLRT for the multi-soliton detection problem, we are again faced with the need for an ML estimate of the positions, $\hat{\underline{\delta}}_{ML}$. A standard approach would involve turning the current problem into one with hypotheses H_0 , H_1 , and H_2 as before, and an additional M hypotheses—one for each value of the parameter $\underline{\delta}$ sampled over a range of possible values. Additionally, the complexity of the detection problem increases exponentially with the number of component solitons, N_s , resulting in a hypothesis testing problem with $(M + 1)^{N_s}$ hypotheses.

Fortunately, as with the estimation problems, the detection problems can be decoupled by preprocessing the signal $r(t)$ with the Toda lattice. If the component solitons separate as viewed on the N th node in the lattice, the detection problem can be more simply formulated using the signal $i_N(t)$. Again, the invertibility of the lattice equations implies that the GLRT decision based on $r(t)$ must be the same as that based on $i_N(t)$. In the high signal-to-noise ratio limit, the noise component of the signal is essentially low pass and Gaussian. Therefore, the GLRT based on the signal $i_N(t)$ reduces to a simple form involving only N_s matched filters.

3.1. Simulations

For simplicity, we consider a hypothesis test between H_0 and H_{12} , where the separation of the two solitons, $\delta_1 - \delta_2$, varies randomly in the interval $[-1/\beta_2, 1/\beta_2]$. The detection processor comprises a Toda lattice of $N = 20$ nodes, with the detection performed based on the signal $i_{10}(t)$. To implement the GLRT, we search over a fixed time interval about the expected arrival time for each soliton and select the maximum matched filter output for each. In this manner we obtain a sequence of 1000 Monte-Carlo values of the processor output for each soliton under each hypothesis. A set of Monte-Carlo runs has been completed for each of 3 different levels of the noise power, N_0 .

The receiver operating characteristic (ROC) for the soliton with $\beta_2 = \sinh(1.5)$ is shown in Fig. 6. For comparison, we also show the ROC that would result from a detection of the soliton alone at the same noise level and with the time-of-arrival known. The detection index, $d = \sqrt{E/N_0}$, is indicated for each case, where E is the energy in the component soliton. The corresponding results for the larger soliton are qualitatively similar, although the detection indices for the soliton with $\beta_1 = \sinh(2)$ alone are 5.6, 4, and 3.3 respectively, therefore the detection probabilities are considerably higher for a fixed probability of false alarm. Note that the detection performance for the small soliton is well modeled by the theoretical performance for detection of the smaller soliton alone.

4. CONCLUSIONS

In this paper we have developed algorithms for parameter estimation and detection of soliton signals in the presence of additive white Gaussian corruption. Each of these algorithms exploits the Toda lattice as a tuned receiver for soliton signals, naturally decoupling the component solitons as they propagate. At high signal to noise ratios, the noise component of the solution to the lattice equations remains low pass and Gaussian and is decoupled from the solitons. This allows for maximum likelihood time-delay estimation and GLRT detection to be performed after preprocessing of the received signal. The resulting estimation algorithms are unbiased and asymptotically approach the Cramér-Rao bounds.

One outstanding issue is the determination of appropriate bounds on the theoretical performance of the GLRT for the detection of soliton and multi-soliton signals. Another potentially interesting area of research includes a theoretical investigation of the performance of the inverse scattering based algorithms for parameter estimation. The lack of a sufficient statistic for estimation of the parameter β stems

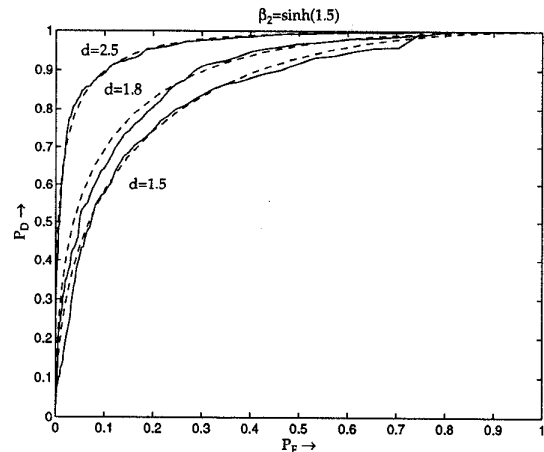


Figure 6. A set of empirically generated ROCs are shown for the detection of soliton 2 under H_{12} . The theoretical ROCs for detection of the soliton under H_2 are indicated with dashed lines, along with the detection indices, d .

from the highly nonlinear manner in which the parameter appears both in the time-scale and the amplitude scale of the signal. Such signals have attracted increasing attention in the research literature, and have been shown to lend themselves naturally to multi-scale processing.

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