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Modular Filters Based on Filter Sharpening and Optimal Minimax Design

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Abstract—**A filter design algorithm is presented that can be viewed as a generalization of filter sharpening with guaranteed minimax optimality and that leads to an efficient modular topology. The structure consists of repetitive usage of a given sub-filter in a fashion similar to a traditional tapped-delay line. In cases where a sub-filter is not specified a priori, a low order sub-filter that approximates the given filter specifications can be used in this structure. The transfer functions of the overall modular filters obtained with this algorithm can be expressed mathematically as the functional composition of two transfer functions. This mathematical formulation creates a convenient framework to analyze and reduce sensitivity with respect to coefficients of the sub-filter or the tap coefficients without altering the characteristics of the overall design.**

I. Introduction

The development of VLSI technology has reduced the emphasis on minimizing the number of multiplications and the number of delay elements in designing filters and has caused a shift toward structures characterized by concurrency [1]. Moreover, VLSI designers are increasingly advocating modularity and regularity in their designs, for example by dividing the overall system into either identical or few distinct sub-systems. This strategy has the advantage of a reduced number of different designs as well as the possibility of independent and efficient verification of sub-systems [2].

In this paper, we present a filter design algorithm that incorporates the desirable properties of modularity. It is based on interconnecting through additions and gains copies of a given low order, easily implementable and computationally efficient sub-filter $G(z)$ in order to obtain a sophisticated overall filter $H(z)$ with sharper frequency response characteristics than $G(z)$. Our approach can be viewed as a generalization of and a more formal approach to filter sharpening as has been considered by several authors including Tukey [3] and Kaiser and Hamming [4]. In our approach, minimax optimality guarantees can be established. The resulting overall design $H(z)$ has a regular structure similar to that of a traditional tapped delay line with the delay elements replaced by the given sub-filter $G(z)$. The proposed design procedure yields the optimal tap parameters f_k, $k = 0, 1, \ldots, K$ where K is the length of the tapped line, which also is the order of desired sharpening in the context of filter sharpening. The parameter K is either pre-specified or determined by hardware constraints. The overall design procedure can be viewed as embedding the sub-filter $G(z)$ in an FIR filter $F(z)$, where $F(z)$ is implemented as a tapped delay line as illustrated in Figure 1.

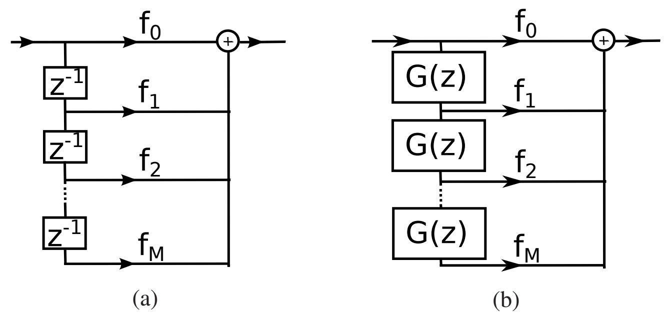


Fig. 1: (a) An FIR filter $F(z) = \sum_{k=0}^{K} f_k z^{-k}$ implemented using a tapped delay line. (b) Embedding $G(z)$ into $F(z)$ to obtain $H(z) = \sum_{k=0}^{K} f_k G^k(z)$.

In cases where a sub-filter is not pre-specified, we propose designing a relatively high order filter $H(z)$ in two steps, where the second step consists of applying sharpening to a low order approximation $G(z)$ obtained in the first step. This approach yields a desirable modular design. Furthermore, it can exhibit much sharper frequency response characteristics with the same number of distinct multipliers than that of a traditional design when the number of multipliers for the low order approximation $G(z)$ and the number of tap coefficients are chosen close to each other. One disadvantage of this approach is an increase in the total number of multiplications compared to a traditional design, although with many current technologies, minimizing the number of multiplications can be less critical than other metrics such as power consumption, regularity and modularity.

When the presented filter design approach is viewed as a generalization of filter sharpening as proposed by Tukey [3], Kaiser and Hamming [4], $G(z)$ in Figure 1 corresponds to the filter to be sharpened and $F(z)$ to the amplitude change function. The z-transform of the overall filter $H(z)$ can be conveniently expressed as the functional composition of a polynomial $F(z)$ and a polynomial or a rational function $G(z)$ in z^{-1}. A formal mathematical framework of polynomial composition can be exploited to find equivalent compositions and improve robustness of $H(z)$ to coefficient quantization and perturbation without changing its frequency domain characteristics when both F and G are polynomials [5].

In Section II, the proposed minimax modular filter design algorithm is presented as a generalization of filter sharpening and the procedure to obtain the optimal tap coefficients f_k is described. The methodology is extended in Section III to designing higher order filters where a sub-filter to be sharpened is not necessarily pre-specified. In Section IV, the properties of modular filters obtained using the proposed method are discussed. In Section V, the mathematical representation of these modular filters as the functional composition of two transfer functions is exploited to reduce sensitivity with respect to their coefficients in the special case when both transfer functions are polynomials.

II. Optimal Filter Sharpening

There have been numerous ad hoc approaches to sharpening a specified filter with frequency response $G(e^{j\omega})$ in order to obtain an overall magnitude response that has smaller deviations from unity in its passband and from zero in its stopband [3], [4]. The simplest approach is to cascade the filter with itself to obtain a response $G^2(e^{j\omega})$, but this has an adverse effect in the passband since squaring will increase the deviation from unity. Tukey proposed a method called *twicing* which involves filtering the input with $G(e^{j\omega})$ and adding back to the input the residual between the input and the output before a second stage of filtering. The effective frequency response in this case becomes

$$\begin{aligned} H_{\text{tw}}(e^{j\omega}) &= (1 + (1 - G(e^{j\omega})))G(e^{j\omega}) \\ &= 2G(e^{j\omega}) - G^2(e^{j\omega}) \end{aligned} \tag{1}$$

Kaiser and Hamming [4] observed that the effective transformation $2x - x^2$ that is being applied to $G(e^{j\omega})$ in this case has a desirable attenuating effect on the passband deviations from unity but an undesirable magnification effect on stopband deviations from zero, the exact opposite effects observed with the transformation x^2. They explained the effect of these transformations through the value of their slope at $x = 0$ for stopband and $x = 1$ for passband; a zero slope will attenuate the magnitude of deviations and a slope that is greater than unity will increase the deviations. Therefore, they proposed using transformations $A(x)$, or *amplitude change functions*, with vanishing first (and also higher order, if desired) derivatives at both $x = 0$ and $x = 1$ in addition to the constraint $A(0) = 0$ and $A(1) = 1$, which guarantees mapping the stopband to zero and the passband to unity. For example, the smallest order polynomial transformation satisfying these constraints is $3x^2 - 2x^3$.

Although Kaiser and Hamming's approach is more systematic than the previous attempts, it is not efficient for sharpening filters that have large deviations from the desired values at the passband and stopband as the methodology assumes only small deviations and ignores higher order terms in the corresponding Taylor series. Moreover, their technique assumes that the filter impulse response is real valued. Therefore its applicability is quite restricted. Even in that case, no optimality guarantees can be established with respect to the Chebyshev or $\mathcal{L}_2$ norms as to whether there exist other amplitude change functions of the same order that better suppress the deviations.

We propose a more systematic formulation of the filter sharpening problem that leads to an optimality guarantee with respect to the Chebyshev or minimax norm. This approach can be formulated as

$$\begin{aligned} &\underset{\mathbf{f}}{\text{minimize}} && \epsilon \\ &\text{subject to} && \|D(\omega) - \sum_{k=0}^{K} f_k G^k(e^{j\omega})\|_\infty \leqslant \epsilon \end{aligned} \tag{2}$$

where $G(e^{j\omega})$ is the possibly complex frequency response of the filter to be sharpened, K is the maximum allowed degree of the sharpening transformation polynomial and

$$|D(\omega)| = \begin{cases} 1, & \omega \in \Omega_P \\ 0, & \omega \in \Omega_S \end{cases} \tag{3}$$

where Ω_P and Ω_S are the passband and stopband of $G(e^{j\omega})$, respectively. This formulation is a special case of a class of semi-infinite optimization problems (SIP) stated as

$$\begin{aligned} &\underset{\mathbf{f}}{\text{minimize}} && \epsilon \\ &\text{subject to} && \|D(\omega) - \sum_{k=0}^{K} f_k U_k(\omega)\|_\infty \leqslant \epsilon \end{aligned} \tag{4}$$

where $U_k(\omega)$, $k = 0, 1, \ldots, K$ are general functions of ω. Algorithms to obtain the optimal coefficients f_k exist when these are all continuous functions of ω. By choosing $U_k(\omega) = G^k(e^{j\omega})$, the resulting optimal coefficients f_k, $k = 0, 1, \ldots, K$ correspond to the tap coefficients in Figure 1 and $H(e^{j\omega})$ becomes the functional composition $F(G(e^{j\omega}))$, where F is the polynomial given by $F(x) = \sum_{k=0}^{K} f_k x^k$.

When $D(\omega)$ has a linear phase and $U_k(\omega) = e^{-jk\omega}$, the optimization problem (4) takes a simpler form that can utilize the alternation theorem and Remez exchange algorithm for its efficient solution [6]. For the same set of functions $U_k(\omega)$ but nonlinear-phase $D(\omega)$, a number of approaches have been developed to solve this optimization problem in the context of complex FIR filter design. These approaches include using linear programming on a dense discrete subset of $[-\pi, \pi]$; generalized Remez algorithms; single or multiple exchange algorithms that solve a discrete sub-problem with optimality guarantees on the entire interval $[-\pi, \pi]$ and Iterative Weighted Least Square, —see Section 2.2 in [7] for a comprehensive overview. Some of these algorithms can be extended to the case of more sophisticated choices for $U_k(\omega)$ than a simple complex exponential as in the case of filter sharpening.

A particularly simple algorithm that is straightforward to extend for general continuous complex basis functions $U_k(\omega)$ is the First Algorithm of Remez [8]. This is a single exchange algorithm which starts with a discrete set of frequencies in the interval $[-\pi, \pi]$, computes the optimal approximating function for those points and repeats the process after adding the frequency of maximum deviation to the next set of constraint points. This procedure is stated in Algorithm 1. At each iteration i, the vector of optimal tap coefficients $\mathbf{f}^{(\mathbf{i})}$ is guaranteed to be in a bounded subset of $\mathcal{R}^{K+1}$, therefore the sequence of vectors $\mathbf{f^{(i)}}$ is guaranteed to have at least one clustering point. Moreover, any of these clustering points will be an optimal choice for the tap coefficients with the same maximum approximation error as the other cluster points [8].

Although the First Algorithm of Remez stated in Algorithm 1 does not theoretically specify a stopping condition for the iterations and requires manually selecting one of the clustering points, our experience suggests that it is sufficient to continue the iterations until the change in the maximum value of the approximation error becomes smaller than a pre-specified tolerance value and then choose the resulting $\mathbf{f^{(i)}}$. This algorithm was used for an assessment of the modular filter design algorithm proposed in this paper due to its simplicity as it only requires solving a finite linear program (or quadratic if $D(\omega)$ and $U_k(\omega)$ are complex) at its first step. More efficient and sophisticated algorithms exist for the solution of semi-infinite optimization problems similar to problem (4).

ALGORITHM 1

Input: $U_k(\omega), k = 0, 1, \ldots, K$ and $D(\omega)$,

Output: $\mathbf{f}^* = \arg\min_{\mathbf{f}} \|D(\omega) - \sum_{k=0}^{K} f_k U_k(\omega)\|_\infty$.

Begin $(i = 1)$

0. Choose $\Omega^{(i)} = \{\omega_0, \omega_1, \ldots, \omega_m\} \subset [-\pi, \pi]$ such that $m \geqslant K$ and the matrix $[U_k(\omega_n)]_{k,n},\ k = 0, 1, \ldots, K$; $n = 0, 1, \ldots, m$ is full rank.

1. Set $\mathbf{f^{(i)}} = \arg\min_{\mathbf{f}} \left\{ \max_{\omega \in \Omega^{(i)}} |D(\omega) - \sum_{k=0}^{K} f_k U_k(\omega)| \right\}$.

2. Find $\omega' = \arg \max_{\omega \in [-\pi,\pi]} |D(\omega) - \sum_{k=0}^{K} f_k^{(i)} U_k(\omega)|$.

3. Set $\Omega^{(i+1)} \leftarrow \Omega^{(i)} \cup \{\omega'\}$ and $i \leftarrow i + 1$, go to Step 1.

The solution for problem (4) is unique if the basis functions $U_k(\omega)$ satisfy the Haar condition [8], [9], in which case Algorithm 1 will converge to the unique optimum [8]. A set of functions $U_k(\omega), k = 0, 1, 2, \ldots, K$ is said to satisfy the Haar condition on $[-\pi, \pi]$ if each $U_k(\omega)$ is continuous and if the matrix

$$V = \begin{bmatrix} U_0(\omega_0) & U_1(\omega_0) & U_2(\omega_0) & \ldots & U_K(\omega_0) \\ U_0(\omega_1) & U_1(\omega_1) & U_2(\omega_1) & \ldots & U_K(\omega_1) \\ U_0(\omega_2) & U_1(\omega_2) & U_2(\omega_2) & \ldots & U_K(\omega_2) \\ & & \vdots & & \\ U_0(\omega_K) & U_1(\omega_K) & U_2(\omega_K) & \ldots & U_K(\omega_K) \end{bmatrix} \quad (5)$$

is full rank for every set $\{\omega_k \in [-\pi, \pi], k = 0, 1, 2, \ldots, K\}$ where all ω_k are distinct. In the context of filter sharpening, $U_k(\omega) = G^k(e^{j\omega})$ where $G(e^{j\omega})$ is the filter to be sharpened. Therefore the matrix V will be a Vandermonde matrix which is always full rank if each row is distinct for every choice of $\{\omega_k, k = 0, 1, \ldots, K\}$ in $[-\pi, \pi]$, or equivalently $G(e^{j\omega})$ is invertible on this interval. Although the Parks-McClellan

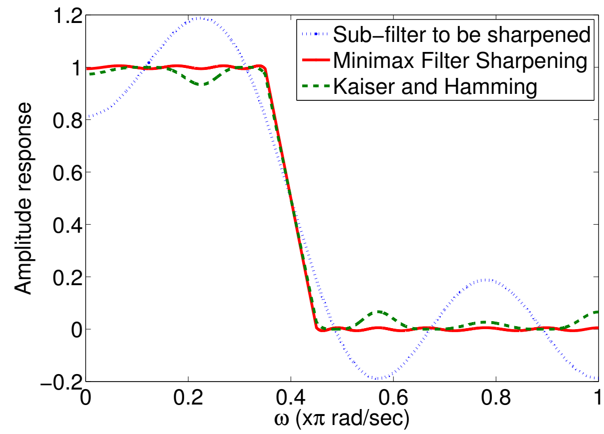


Fig. 2: The frequency response of a 10^{th}-order filter $G(e^{j\omega})$ before and after sharpening with two different 7^{th}-order transformation polynomials. The polynomials were obtained using Kaiser and Hamming's proposed method in [4] and our proposed algorithm.

algorithm can be easily extended to the case where the Haar condition is satisfied by $G^k(e^{j\omega})$, $k = 0, 1, \ldots, K$ and $G(e^{j\omega})$ and $D(\omega)$ are real, neither is required to obtain an optimal solution for equation (2) using Algorithm 1.

Figure 2 illustrates the comparison of frequency responses of a 10^{th} order low-pass filter $G(e^{j\omega})$ obtained using the Parks-McClellan filter design algorithm with $\Omega_P = [0, 0.35\pi]$ and $\Omega_S = [0.45\pi, \pi]$, the response of the sharpened filter with a 7^{th}-order transformation using Kaiser and Hamming's method [4] and the sharpened filter with the optimal 7^{th} order polynomial obtained using minimax filter sharpening algorithm that we propose. This example clearly shows that the proposed optimal approach to the filter sharpening problem yields a better frequency response over the entire interval especially where the sub-filter exhibits large ripples.

III. Two-Step Modular Filter Design

In the design of a linear phase FIR filter $H(e^{j\omega})$, a desirable modular structure similar to the one in Figure 1 can be obtained by seeking an approximation of the form $F(G(e^{j\omega}))$ such that the maximum approximation error $\|H(e^{j\omega}) - F(G(e^{j\omega}))\|_\infty$ is minimized. The problem of finding the best approximation to a real and continuous function over an interval using polynomial composition of given orders has been addressed in [10]. However, it was shown that counting the alternations of the approximation error in general does not characterize the best approximation using composition of polynomials unlike the case of a single polynomial as the best approximation which can be identified using the alternation theorem. This task is more formidable when the function to be approximated is complex and the desired orders of the composing polynomials are not pre-specified. Therefore, we depart to a heuristic and greedy two-step approach to obtain such modular filter structures consisting of obtaining a sub-

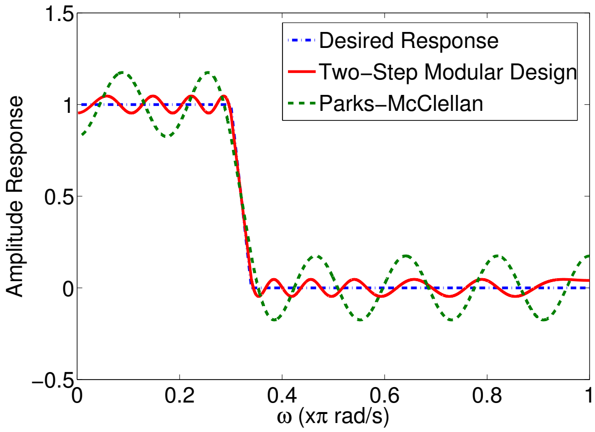
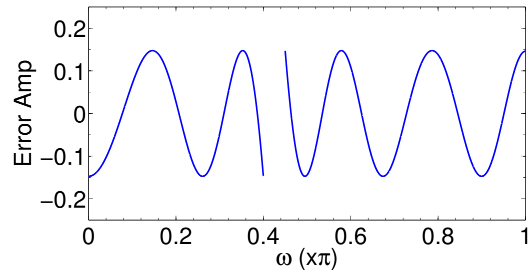


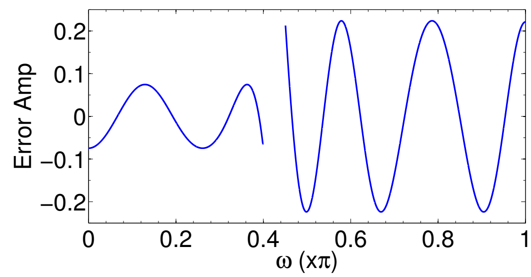
Fig. 3: The comparison of a 24^{th}-order Parks-McClellan filter and a modular filter obtained using the two-step filter design algorithm with a 7^{th}-order polynomial and a 10^{th}-order Parks-McClellan filter. Both designs can be shown to have the same number of distinct multipliers while the modular design has superior frequency response characteristics.

filter G and then an optimal F rather than jointly optimizing them. This algorithm starts with splitting the allowed filter order P, which can be viewed as the number of available distinct multipliers or degrees of freedom, between these two components. The sub-filter is designed as the optimal zero-phase FIR filter with its N allocated degrees of freedom, and the remaining $M = P - N$ degrees of freedom are used to choose the coefficients of F to improve the frequency response characteristics of $G(e^{j\omega})$ using Algorithm 1, which yields the optimal choice since the sub-filter is already determined. In our simulations, the best results using this heuristic algorithm seem to be obtained when the degrees of freedom for F and $G(e^{j\omega})$ are chosen close to each other for low-pass and high-pass linear phase FIR filters.

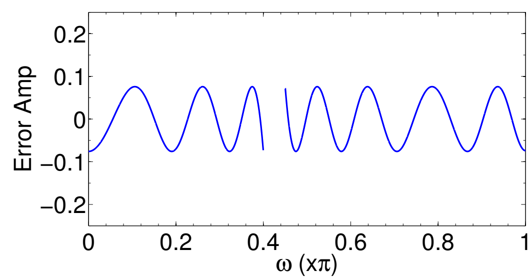
In Figure 3, a 24^{th}-order zero-phase Parks-McClellan low-pass filter with passband $[0, 0.30\pi]$ and stopband $[0.34\pi, \pi]$ is compared to a composition of a 7^{th} order polynomial F and a 10^{th} order sub-filter $G(e^{j\omega})$, where $G(e^{j\omega})$ is also a Parks-McClellan filter with the same passband and stopband edge frequencies as the 24^{th} order filter. Even-order Parks-McClellan filters can be implemented using distinct coefficients as many as one plus half of their order due to their symmetry, therefore both of these designs can be shown to be implemented using the same number of distinct multipliers. The compositional design yields a considerably superior frequency response characteristics. In general, the total number of multiplications in a compositional design of Figure 1 is greater than a direct form implementation with the same number of distinct multipliers, $M + N + 1$. However, the compositional design has a desirable modular structure and a minor increase in the number of multiplications is often not the primary concern in the current VLSI design technology [1].



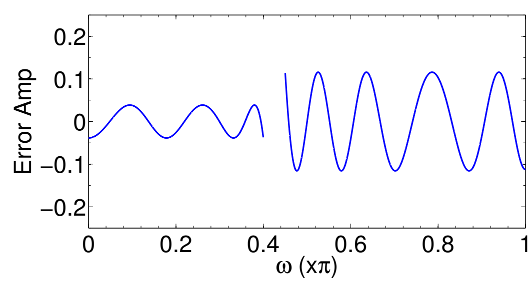
(a)



(b)



(c)



(d)

Fig. 4: The approximation errors to an ideal low pass filter with passband and stopband edge frequencies 0.40π and 0.45π, respectively. A low (10^{th})order filter $G(e^{\,})$ with the same passband and stopband edges is used repetitively in tapped line with tap coefficients $f_k, k = 0, 1, \ldots, M$: (a) $M = 4$ with uniform weight. (b) $K = 4$ with relative passband error weight of three. (c) $M = 6$ with uniform weight. (b) $K = 6$ with relative passband error weight of three.

As an example of the convenience of modularity that results from designing filters using the proposed two-step design procedure, consider that a low order low pass filter $G(e^{j\omega})$

is obtained in the first step. A sharper low pass filter with the same passband and stopband edges can be obtained by choosing appropriate tap coefficients $f_k, k = 0, 1, \ldots, M$ that will minimize the maximum deviation from the desired response. If the need to have sharper characteristics in one of the bands than the other band arises during an application or equivalently an explicit weight function is specified, it will suffice to re-compute the tap coefficients consistent with the weight function without altering the filter $G(e^{j\omega})$. If even sharper characteristics are desired in both bands, then new blocks of $G(e^{j\omega})$ and their corresponding tap coefficients can simply be appended to the tapped structure to increase the overall filter order. Figure 4 illustrates several different error functions resulting from approximating an ideal low pass filter with a passband edge 0.40π and a stopband edge of 0.45π. In Figure 4a, the tap coefficients $f_k, k = 0, \ldots, 4$ are chosen to minimize the maximum error using four blocks of $G(e^{j\omega})$, where $G(e^{j\omega})$ is a 10^{th} order low pass filter with a passband edge 0.40π and a stopband edge of 0.45π. In the case where the passband error is assigned to have a weight of three times that of the error in the stopband, the tap coefficients $f_k, k = 0, 1 \ldots, 4$ can be changed to obtain the error profile given in Figure 4b. By allowing the use of two more blocks of $G(e^{j\omega})$, the tap coefficients $f_k, k = 0, 1, \ldots, 6$ can be chosen to obtain smaller errors with similar weight functions as illustrated in Figure 4c and 4d. Although in these examples the weighted errors have equi-oscillatory behavior, this is not necessarily the case for general frequency responses and weight functions as best approximation by polynomial composition is not in general characterizable by equi-oscillations [10].

IV. Properties of Modular Filters

The frequency response of a modular filter $H(e^{j\omega})$ can be expressed as the functional composition of a polynomial F with another polynomial or a rational function $G(e^{j\omega})$ in $e^{j\omega}$. This can be explicitly stated as

$$H(e^{j\omega}) = f_0 + f_1 G(e^{j\omega}) + f_2 G^2(e^{j\omega}) + \cdots + f_K G^K(e^{j\omega}) \quad (6)$$

and corresponds to the impulse response

$$h_n = f_0 + f_1 g_n + f_2(g_n * g_n) + \cdots + f_K(g_n * \cdots * g_n) \quad (7)$$

where "$*$" corresponds to convolution of discrete time sequences. From equation (7), it follows that the filter $H(e^{j\omega})$ obtained as the composition of a polynomial F and a causal filter $G(e^{j\omega})$ is also causal. Stability of the composition can be deduced similarly when $G(e^{j\omega})$ is stable since F has a finite order.

An important and desirable property for filters is that they not introduce dispersion, i.e. they have zero or linear phase. Linear-phase FIR filters can be obtained by composing any polynomial F with a zero phase filter $G(e^{j\omega})$, the impulse response of which is even-symmetric around zero. Each term in equation (7) can be shown to be even-symmetric around zero and hence the over all impulse response. The filter can be cascaded with delays appropriately to obtain a causal and linear-phase filter with the same magnitude response. For $G(e^{j\omega})$ with odd-symmetric impulse responses, the composed filter obtained this way will be linear phase if either all odd-indexed coefficients or all even-indexed coefficients are zero. This follows from the fact that an odd number of self convolutions of the sequence g_n are odd-symmetric and an even number of self convolutions are even symmetric; and in general sum of odd symmetric and even symmetric sequences not necessarily remain symmetric in either type.

For modular filters obtained as a composition as in Figure 1, the design effort has to be allocated only for one block, namely $G(e^{j\omega})$. This could be argued to possibly induce a sensitivity problem since an error in the design of $G(e^{j\omega})$ such as a perturbation in is coefficients could significantly affect the overall behavior of the filter. However, the compositional form of the transfer function $H(e^{j\omega}) = F(G(e^{j\omega}))$ offers the flexibility to choose an equivalent design with much less sensitivity to the coefficients of either F or G. This is particularly easy to analyze in the case where $G(e^{j\omega})$ is FIR as described in the next section.

V. Coefficient Sensitivity

Sensitivity of the frequency response of $H(e^{j\omega})$ to the perturbations in the coefficients of F and $G(e^{j\omega})$ can be defined as [5]

$$S_{U\to H} = \max_{\Delta\mathbf{u}} \frac{E_{\Delta\mathbf{h}}/E_{\mathbf{h}}}{E_{\Delta\mathbf{u}}/E_{\mathbf{u}}} \quad (8)$$

where U is either F or G depending on which is being perturbed, $\mathbf{u}$ is the coefficient vector of U, $\Delta\mathbf{u}$ is an infinitesmall perturbation vector, $\|\cdot\|$ indicates l_2 norm and

$$E_{\mathbf{u}} = \int_{-\pi}^{\pi} |U(e^{j\omega})|^2 \, d\omega \quad (9)$$

or equivalently

$$E_{\mathbf{u}} = \mathbf{u}^{\mathbf{T}}\mathbf{u} = \|\mathbf{u}\|_2^2. \quad (10)$$

This sensitivity corresponds to the worst case amplification of a perturbation in the coefficients of F or G. When G is an FIR filter, the sensitivities can be obtained in closed form using their coefficients. Note that equation (7) suggests a linear relationship between $\mathbf{f}$ and $\mathbf{h}$, the coefficient vectors of F and H, given by

$$\mathbf{h} = \mathbf{C}\mathbf{f} \quad (11)$$

where the kth column of matrix $\mathbf{C}$ consists of $(k-1)$ self-convolutions of g_n.

A. Formulation of $S_{F\to H}$

A perturbation $\Delta\mathbf{f}$ in $\mathbf{f}$ will result in a change in $\mathbf{h}$ given by

$$\Delta\mathbf{h} = \mathbf{C}\Delta\mathbf{f}. \quad (12)$$

The sensitivity of the composed filter with respect to F becomes, using equation (8), (11) and (12),

$$S_{F\to H} = \max_{\Delta\mathbf{f}} \frac{||\mathbf{C}\Delta\mathbf{f}||_2^2}{||\Delta\mathbf{f}||_2^2} \frac{||\mathbf{f}||_2^2}{||\mathbf{C}\mathbf{f}||_2^2}. \quad (13)$$

For a given F and G, the factor $\frac{||\mathbf{f}||_2^2}{||\mathbf{Cf}||_2^2}$ is constant. The maximum value of $\frac{||\mathbf{C}\Delta\mathbf{f}||_2^2}{||\Delta\mathbf{f}||_2^2}$ is equal to $\sigma^2_{\mathbf{C},max}$, where $\sigma_{\mathbf{C},max}$ is the maximum singular value of $\mathbf{C}$. Therefore equation (13) becomes

$$S_{F\to H} = \sigma^2_{\mathbf{C},max}\frac{||\mathbf{f}||_2^2}{||\mathbf{Cf}||_2^2}. \tag{14}$$

B. Formulation of $S_{G\to H}$

The relationship between the coefficient vectors $\mathbf{g}$ and $\mathbf{h}$ is nonlinear, however in the case of infinitesmall perturbations $\Delta\mathbf{g}$, we have [5]

$$\Delta\mathbf{h} = \mathbf{D}\Delta\mathbf{g} \tag{15}$$

where $\mathbf{D}$ is an $(MN+1)\times(N+1)$ Toeplitz matrix the first column of which consists of the coefficients of the polynomial $D(x)$ with zero padding of length N. Here M and N are the orders of F and G, respectively, and

$$D(x) = F'(G(x)). \tag{16}$$

The sensitivity of the composed filter with respect to G becomes, using equations (8) and (15),

$$S_{G\to H} = \max_{\Delta\mathbf{g}}\frac{||\mathbf{D}\Delta\mathbf{g}||_2^2}{||\Delta\mathbf{g}||_2^2}\frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2}. \tag{17}$$

As in the previous section, for a given F and G, $\frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2}$ is constant. The maximum value of $\frac{||\mathbf{D}\Delta\mathbf{g}||_2^2}{||\Delta\mathbf{g}||_2^2}$ is $\sigma^2_{\mathbf{D},max}$, where $\sigma_{\mathbf{D},max}$ is the maximum singular value of $\mathbf{D}$. Therefore equation (17) becomes

$$S_{G\to H} = \sigma^2_{\mathbf{D},max}\frac{||\mathbf{g}||_2^2}{||\mathbf{h}||_2^2}. \tag{18}$$

C. Equivalent low-sensitivity representations

The convenient compositional representation of modular filters allows to obtain infinitely many equivalent compositions of the form

$$F(G(z)) = (F\circ\lambda^{-1})\circ(\lambda\circ G)(z) = \bar{F}(\bar{G}(z)), \tag{19}$$

where $\lambda(z) = az+b$ is any first order polynomial. It has been shown that by careful choice of a and b in λ, an identical frequency response $\bar{F}(\bar{G}(e^{j\omega}))$ with much lower coefficient sensitivities can be obtained [5].

VI. Conclusion

A modular filter design algorithm that can be viewed as a generalization of and a more systematic approach to filter sharpening is presented. Sharpened filters are shown to have transfer functions in the form of a polynomial composed with the transfer function of the sub-filter to be sharpened. The algorithm yields the coefficients of a polynomial that has minimax optimality guarantees in that no other polynomial of the same order suppresses better the deviations in the passband and the stopband. In an implementation, the coefficients of the polynomial correspond to the tap coefficients in a structure similar to a tapped delay line. The algorithm can be generalized to cases where a sub-filter is not pre-specified, and sharper characteristics than traditional designs can be obtained using the same number of degrees of freedom to design the overall filter. A convenience of modularity is shown where it allows to change error weights in different bands by simply changing the tap coefficients and not altering the basic sub-filter in the overall implementation. Finally, a property of polynomial composition is exploited to reduce sensitivity of the modular filters with respect to perturbations in the sub-filter coefficients or the tap coefficients without changing the overall frequency response.

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