Systematic Synthesis Procedures for High-Dimensional Chaotic Systems

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ABSTRACT

In practical applications of self-synchronizing chaotic systems, it is undesirable to be restricted to the well-known low-dimensional systems; we need the ability to choose from a wide variety of high-dimensional chaotic systems. This paper discusses systematic synthesis procedures for achieving this capability. The significance of this work lies in the fact that the ability to synthesize high-dimensional chaotic systems further enhances their applicability for communications, signal processing, and modeling of physical processes.

I. Introduction

The chaotic self-synchronization property [1, 2] suggests some intriguing concepts for embedding information within a chaotic transmission and for recovering the message at the intended receivers [3, 4, 5]. The current challenge in this area is to create high-dimensional self-synchronizing chaotic systems which are practical to implement and can be applied and tested against real-world problems. Previous developments of systematic synthesis procedures have addressed this issue [6, 7]; the procedures provided a means for designing a wide variety of high-dimensional chaotic systems that can be implemented with standard analog hardware. The many possible system designs improve the privacy aspects of self-synchronizing chaotic systems.

In [6], Lyapunov's direct method was used as a basis to develop the systematic synthesis procedure for a general class of quadratically nonlinear chaotic systems that synchronize via a single drive signal. While those systems appear to be very promising, the state equations may contain a large number of nonlinear terms making their circuit implementation potentially difficult. The desire to create high-dimensional chaotic systems that are easier to implement motivated a later study of mutually coupled chaotic systems [7]. A systematic synthesis capability was developed for a class of chaotic arrays which possess the self-synchronization property. Chaotic arrays offer considerable flexibility in the design of complex chaotic systems, although they typically require multiple drive signals for synchronization - - significantly increasing the complexity of the communication system.

The work presented here focuses on an interesting subclass of chaotic arrays - - those consisting of a single low-dimensional chaotic system and an N-dimensional linear feedback system. These systems, which we refer to as "Linear FeedBack Chaotic Systems (LFBCSs)," have several implementation advantages. They provide for high design flexibility and suggest straightforward circuit implementations.

The paper is organized as follows. In Section II, we describe the synthesis methodology used to develop the various synthesis procedures. In Section III, we demonstrate the synthesis approach for LFBCSs. Section IV summarizes the results of this work.

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II. Synthesis Methodology

Our approach to synthesis has consistently followed a systematic four step process. First, we specify an algebraic model for the transmitter and receiver systems. As shown in [6, 7], the chaotic system models can be very general; in [6] the model represents a large class of quadratically nonlinear systems, while in [7] the model allows for an unlimited number of Lorenz oscillators to be mutually coupled via an N-dimensional linear system.

The second step in the synthesis process involves subtracting the receiver equations from the transmitter equations and imposing a global asymptotic stability constraint on the resulting error equations. Using Lyapunov's direct method, sufficient conditions for the error system's global stability are usually straightforward to obtain. The sufficient conditions determine constraints on the free parameters of the transmitter and receiver which guarantee that they possess the global self-synchronization property.

The third step in the synthesis process focuses on the global stability of the transmitter equations. First, a family of ellipsoids in state space is defined and then sufficient conditions are determined which guarantee the existence of a *trapping region.* The trapping region imposes additional constraints on the free parameters of the transmitter and receiver equations.

The final step involves determining sufficient conditions which render all of the transmitter's fixed points unstable. In most cases, this involves numerically integrating the transmitter equations and computing the system's Lyapunov exponents and/or attractor dimension [8]. If stable fixed points exist, the system's bifurcation parameter is adjusted until they all become unstable. In the next section, we demonstrate the synthesis approach for LFBCSs.

III. Synthesizing Self-Synchronizing Chaotic Systems

Linear feedback chaotic systems (LFBCSs) are composed of a low-dimensional chaotic system and a linear feedback system as illustrated in Fig. 1. Because the linear system is N-dimensional, considerable design flexibility is possible with LFBCSs. Another practical property of LFBCSs is that they synchronize via a single drive signal while exhibiting complex dynamics.

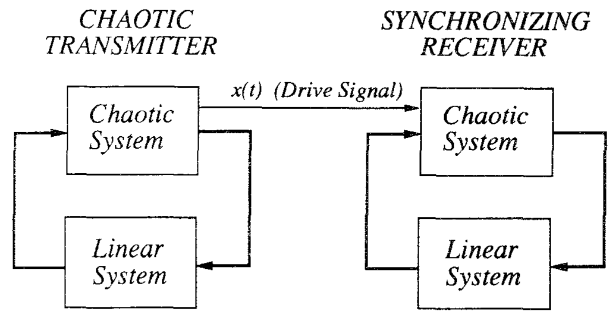

Figure 1: *Linear Feedback Chaotic Systems.*

While many types of LFBCSs are possible, we considered two specific cases: *(i)* the chaotic Lorenz signal $x(t)$ drives an N-dimensional linear system and the output of the linear system is added to the equation for $\dot{x}$ in the Lorenz system; and *(ii)* the Lorenz signal $z(t)$ drives an N-dimensional linear system and the output of the linear system is added to the equation for $\dot{z}$ in the Lorenz system. In both cases, a complete synthesis procedure was developed.

To illustrate the approach, consider the x-input/x-output case, *i.e.*, where the transmitter equations are given by

$$\begin{aligned} \dot{x} &= \sigma(y-x)+\nu \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \\ \dot{\mathbf{l}} &= A\mathbf{l} + Bx \\ \nu &= C\mathbf{l} + Dx \ . \end{aligned} \tag{1}$$

The first three state equations correspond to the Lorenz system. The vector $\mathbf{l}$ and scalar ν denote the state variables and output of the linear system, respectively. The linear system is N-dimensional, *i.e.*, A is $N \times N$, B is $N \times 1$, C is $1 \times N$, and D is 1×1. For notational simplicity, we refer to the transmitter state variables collectively by the vector $\mathbf{x} = (x, y, z, \mathbf{l})$, when convenient.

The self-synchronization properties of the Lorenz system suggested a receiver system of the form

$$\begin{aligned}\dot{x}_r &= \sigma(y_r - x_r) + \nu_r \\ \dot{y}_r &= rx(t) - y_r - x(t)z_r \\ \dot{z}_r &= x(t)y_r - bz_r \\ \dot{\mathbf{l}}_r &= A\mathbf{l}_r + Bx(t) \\ \nu_r &= C\mathbf{l}_r + Dx(t) \ .\end{aligned} \tag{2}$$

Algebraically, the receiver system (2) is obtained from the transmitter (1) by renaming variables $\mathbf{x} \to \mathbf{x}_r$ and substituting the drive signal $x(t)$ for $x_r(t)$ in all state equations except the first.

We can study the self-synchronization properties of the transmitter and receiver equations by forming the error system. The error system is derived by subtracting (2) from (1) to obtain

$$\begin{aligned}\dot{e}_x &= \sigma(e_y - e_x) + C\mathbf{e}_l \\ \dot{e}_y &= -e_y - x(t)e_z \\ \dot{e}_z &= x(t)e_y - be_z \\ \dot{\mathbf{e}}_l &= A\mathbf{e}_l \ .\end{aligned} \tag{3}$$

Since the dynamics of $\mathbf{e}_l$ are independent of e_x, e_y, and e_z, we can see that if A is a stable matrix, then the $\mathbf{e}_l$ subsystem is globally asymptotically stable at the origin. The (e_y, e_z) subsystem is also decoupled from the rest of the system, and can be shown to be globally asymptotically stable at the origin [4]. The error signal $e_x(t)$ must also go to zero as $t \to \infty$ because $e_x(t)$ is the output of a stable linear time-invariant system that is driven by $e_y(t)$ and $\mathbf{e}_l(t)$. From this analysis, we conclude that the error system is globally asymptotically stable at the origin if A is a stable matrix. Equivalently, with A as a stable matrix, the transmitter and receiver are guaranteed to synchronize regardless of the initial conditions imposed on these systems.

The next step is to determine an appropriate set of conditions which guarantee that the transmitter is globally stable. The global stability conditions, together with the self-synchronization conditions, suggest a systematic synthesis procedure. Below, we summarize the procedure; a complete development is given elsewhere [9].

Synthesis Procedure

1. Choose any stable A matrix and any $N \times N$ symmetric positive definite matrix Q.
2. Solve $PA + A^T P + Q = 0$ for the positive definite solution P.
3. Choose any vector B and set $C = -B^T P/r$.
4. Choose any D such that $\sigma - D > 0$.

The first step of the procedure is simply the self-synchronization condition; it requires the linear system to be stable. Clearly, many choices for A are possible. The second and third steps are akin to a negative feedback constraint, *i.e.*, the linear feedback tends to stabilize the chaotic system. The last step in the procedure restricts $\sigma - D > 0$ so that the $\dot{x}$ equation of the Lorenz system remains dissipative after feedback is applied.

For the purpose of demonstration, consider the following five-dimensional x-input/x-output LFBCS.

$$\begin{aligned}\dot{x} &= \sigma(y-x) + \nu \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \\ \begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2} & 10 \\ -10 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x \\ \nu &= -\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\end{aligned} \tag{4}$$

It can be shown in a straightforward way that the linear system satisfies the synthesis procedure for suitable choices of P, Q, and R. For the numerical demonstrations presented below, the Lorenz parameters chosen are $\sigma = 16$ and $b = 4$; the bifurcation parameter r will be varied.

In Fig. 2, we show the computed Lyapunov dimension as r is varied over the range, $20 < r < 100$. This figure demonstrates that the LFBCS achieves a greater Lyapunov dimension than the Lorenz system without feedback. The Lyapunov dimension could be increased by using more states in the linear system. However, numerical experiments suggest that stable linear feedback creates only negative Lyapunov exponents, limiting the dynamical complexity of LFBCSs. Nevertheless, their relative ease of implementation is an attractive practical feature.

In Fig. 3, we demonstrate the rapid synchronization between the transmitter and receiver systems. The curve measures the distance in state space between the transmitter and receiver trajectories when

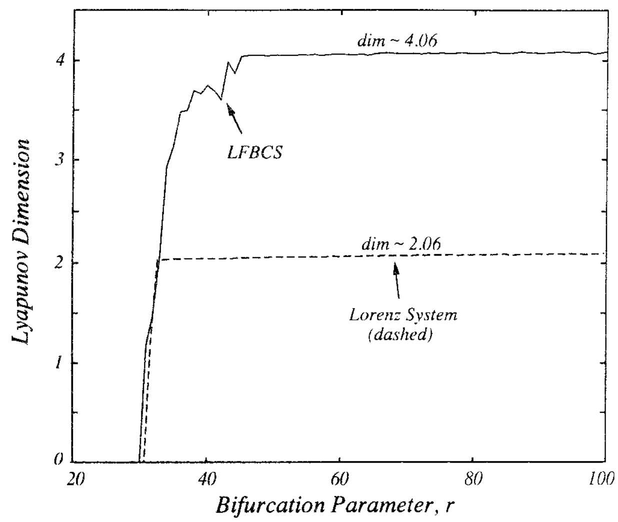

Figure 2: *Lyapunov Dimension of a 5-D LFBCS.*

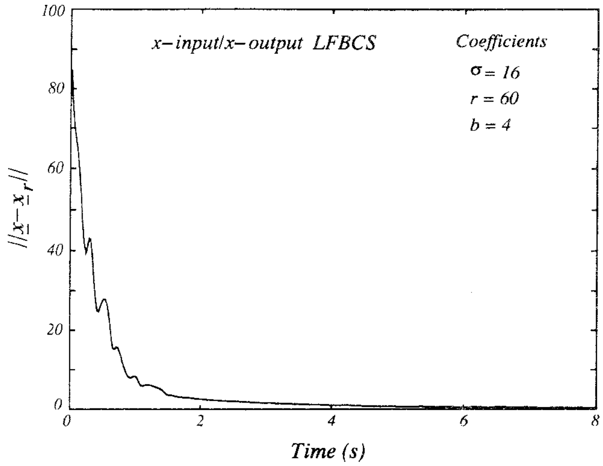

Figure 3: *Self-Synchronization in a 5-D LFBCS.*

the receiver is initialized from the zero state. Synchronization is maintained indefinitely.

IV. Conclusions

The synthesis methodology described in this paper provides the potential for designing high-dimensional self-synchronizing chaotic systems which could be implemented in hardware and used in practice. The linear feedback chaotic system concept described here provides for high design flexibility and suggests straightforward circuit implementations. While these results appear promising, much work remains before chaotic communication systems can be considered truly private or practical. We are currently exploring the applied aspects of these systems.

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