COMPENSATION OF COEFFICIENT ERASURES IN FRAME REPRESENTATIONS

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ABSTRACT

This work explores low complexity systems to compensate for coefficient erasures in frame representations of signals. Assuming linear synthesis with a pre-specified frame it is demonstrated that erasures can be compensated for even if the origin of the representation coefficients is not known. If the transmitter is aware of the erasure occurrence, the compensation is performed by projecting the erasure error to the remaining coefficients. Furthermore, it is demonstrated that the same compensation can be executed using a transmitter/receiver combination in which the transmitter is not aware of the erasure occurrence. The transmitter compensates for all the coefficients using projections, assuming an erasure will occur. The receiver undoes the compensation for the coefficients that have not been erased, thus maintaining the compensation only of the erased coefficients.

1. INTRODUCTION

In a variety of signal processing and communications contexts, erasures occur inadvertently or can be intentionally introduced as part of a data reduction strategy. Frame representations are generalizations of basis representations providing redundancy and, therefore, robustness to signal degradation such as noise, quantization, and erasures. This paper explores the use of frames and projections to compensate for erasures.

Finite frames represent a vector $\mathbf{x}$ in a space $\mathcal{W}$ of finite dimension N using the synthesis equation:

$$\mathbf{x} = \sum_k a_k \mathbf{f}_k, \tag{1}$$

in which the a_k are the representation coefficients, and the synthesis frame vectors $\{\mathbf{f}_k, k = 1, \ldots, M\}$ span the space $\mathcal{W}$. This condition requires that $M \geq N$. Usually, but not always, the representation coefficients are determined by the analysis equation:

$$a_k = \langle \mathbf{x}, \underline{\mathbf{f}}_k \rangle, \tag{2}$$

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in which the analysis frame vectors $\{\underline{\mathbf{f}}_k, k = 1, \ldots, M\}$ also span $\mathcal{W}$. Details on frame representations and the relationships of the analysis and synthesis vector sets can be found in a variety of texts such as [1]. The ratio $r = M/N$ is referred to as the *redundancy* of the frame. For infinite dimensional spaces, using a frame ensures the sum converges for all $\mathbf{x}$ with finite magnitude.

In contrast to basis representations, frame representations are not unique. Given a synthesis frame $\{\mathbf{f}_k, k = 1, \ldots, M\}$, different sets of coefficients a_k might produce the same vector $\mathbf{x}$ in $\mathcal{W}$, depending on the redundancy of the frame. This property decouples the analysis from the synthesis process. The a_k can be determined in a variety of ways (for some examples, see [2, 3] and references within). The a_k might also be original data to be processed using the synthesis sum (1), not originating from an expansion of $\mathbf{x}$.

Similarly, the representation coefficients a_k of a vector analyzed using the analysis frame and equation (2) can be used in a variety of ways to synthesize the vector. For example, the reconstruction might use only a subset, as long as the corresponding frame vectors span the space, making the representation robust to erasures. Indeed, most of the existing work on erasures on frame representations assumes that $\mathbf{x}$ is expanded using the analysis equation. Using that assumption, the synthesis is modified to reconstruct the original signal. For example, linear reconstruction can be performed using a recomputed synthesis frame and equation (1) [3, 4]. Alternatively the erased coefficients can be re-computed using the non-erased ones, and used to fill in the coefficient stream before the synthesis is performed using linear reconstruction with the original synthesis frame [5, 6]. However, neither approach is possible without assuming an expansion using equation (2).

In our work, rather than assuming that the vector is analyzed using the analysis equation (2), we make no assumptions on how the representation coefficients a_k are generated. We only assume that the synthesis is performed using a pre-specified synthesis frame and the synthesis sum of equation (1). Thus, it is not possible to fill in the missing coefficients or modify the synthesis frame at the receiver. Instead, we modify the representation coefficients at the transmitter to compensate for the erasure. This assumes that the transmitter is aware that an erasure occurs, which is often not the case. Still, even if only the receiver is aware that an erasure occurs, we demonstrate that a simple transmitter/receiver combination can be used to compensate for the erasure using the same orthogonal projection principle. The transmitter modifies the frame representation assuming the erasures will occur, and the receiver undoes the changes if the erasures do not occur. The advantage of this approach is that the representation coefficients may be generated in a variety of ways, including the analysis equation or the use of the matching pursuit [2]. They might also be original data to be processed using the synthesis sum.

The use of projections to compensate for errors has been considered in [7, 8] as an extension of quantization noise shaping to arbitrary frame expansions. However, in that work, the quantization error is known at the transmitter—not necessarily the case with erasure errors. The use of redundancy to compensate for erasures assuming a fixed reconstruction method has also been considered in a different context in [9, 10]. In that work the error is again known at the transmitter, and only the case of LTI reconstruction filters is considered. The problem is formulated and solved as a constrained optimization.

In the next section we state the problem and establish the notation. In section 3 we demonstrate that the optimal solution is the orthogonal projection of the erasure error to the span of the remaining synthesis vectors. A causal implementation is proposed in section 4, assuming the transmitter is aware of the erasure. Section 5 presents a transmitter that pre-compensates for the erasure and a receiver that undoes the compensation if the erasure does not occur. The causality constraints imposed in part of this work, often allow only for a partial compensation of the error.

2. PROBLEM STATEMENT

Consider the synthesis of a vector $\mathbf{x}$:

$$\mathbf{x} = \sum_k a_k \mathbf{f}_k, \tag{3}$$

in which we make no assumptions on how the representation coefficients $\{a_k\}$ originate.

The coefficients $\{a_k\}$ are used to synthesize the signal using the pre-specified synthesis frame $\{\mathbf{f}_k\}$, subject to erasures known at the transmitter or the receiver. We model erasures as replacement of the corresponding a_k with 0, i.e. removal of the corresponding term $a_k\mathbf{f}_k$ from the summation in (3). Since the expansion method is not known, the goal is to compensate for the erasure as much as possible using the remaining non-erased coefficients.

Thru section 4 we assume that the transmitter anticipates an erasure and knows the value of the erased coefficient. Assuming coefficient a_i is erased, the transmitter is only allowed to modify the coefficients $\{a_k|k \in S_i\}$ to $\{\hat{a}_k|k \in S_i\}$ in order to compensate for the erasure. $S_i = \{k_1, \ldots, k_p\}$ denotes the set of coefficient indices used for the compensation of a_i. The reconstruction is performed using equation (3) with the updated coefficients:

$$\hat{\mathbf{x}} = \sum_{k \in S_i} \hat{a}_k \mathbf{f}_k + \sum_{k \notin S_i, k \neq i} a_k \mathbf{f}_k, \tag{4}$$

such that $\hat{\mathbf{x}}$ minimizes the magnitude of the error $\mathcal{E} = \mathbf{x} - \hat{\mathbf{x}}$.

3. COMPENSATION USING PROJECTIONS

The error due to the erasure and compensation can be rewritten using the synthesis sums:

$$\mathcal{E} = a_i \mathbf{f}_i + \sum_{k \in S_i} (a_k - \hat{a}_k) \mathbf{f}_k \tag{5}$$

The vectors $\{\mathbf{f}_k|k \in S_i\}$ span a space $\mathcal{W}_i$. Therefore, the error magnitude is minimized if the sum $\sum_{k \in S_i}(a_k - \hat{a}_k)\mathbf{f}_k$ is the orthogonal projection[1] of $-a_i\mathbf{f}_i$ onto $\mathcal{W}_i$.

Exploiting the linearity of projections, we define the *projection coefficients* $c_{i,k}$ such that they satisfy:

$$\mathcal{P}_{\mathcal{W}_i}(\mathbf{f}_i) = \sum_{k \in S_i} c_{i,k} \mathbf{f}_k, \tag{6}$$

in which $\mathcal{P}_{\mathcal{W}_i}(\mathbf{f}_i)$ is the projection of $\mathbf{f}_i$ onto $\mathcal{W}_i$. Thus, the compensation of the erasure of a_i can be performed optimally by updating each of the a_k to:

$$\begin{aligned} \hat{a}_k &= a_k + a_i c_{i,k}, \text{ for all } k \in S_i & (7) \\ \Rightarrow \mathcal{E} &= a_i \mathbf{f}_i - a_i \sum_{k \in S_i} c_{i,k} \mathbf{f}_k & (8) \\ &= a_i(\mathbf{f}_i - \mathcal{P}_{\mathcal{W}_i}(\mathbf{f}_i)). & (9) \end{aligned}$$

Using inner products with $\{\mathbf{f}_k|k \in S_i\}$ on (6), and defining the frame autocorrelation as $R_{k,l} = \langle \mathbf{f}_k, \mathbf{f}_l \rangle$, the determination of the projection coefficients becomes equivalent to choosing the $c_{i,k}$ as the solution to:

$$\begin{bmatrix} R_{k_1,k_1} & \cdots & R_{k_1,k_p} \\ \vdots & \ddots & \vdots \\ R_{k_p,k_1} & \cdots & R_{k_p,k_p} \end{bmatrix} \begin{bmatrix} c_{i,k_1} \\ \vdots \\ c_{i,k_p} \end{bmatrix} = \begin{bmatrix} R_{i,k_1} \\ \vdots \\ R_{i,k_p} \end{bmatrix}. \tag{10}$$

Satisfying (10) is equivalent to computing the frame expansion of $\mathbf{f}_i$ using $\{\mathbf{f}_k|k \in S_i\}$ as a synthesis frame. If the frame vectors $\{\mathbf{f}_k|k \in S_i\}$ are linearly dependent, the solution to (10) is not unique. All the possible solutions are optimal given the cost function and the constraint that only

[1] Henceforth the term projection refers to *orthogonal* projections.

coefficients $\{a_k | k \in S_i\}$ can be modified. If the vector $a_i \mathbf{f}_i$ being compensated is in the span of the vectors $\{\mathbf{f}_k | k \in S_i\}$ used for the compensation (i.e. $\mathbf{f}_i \in \mathcal{W}_i$), then the erasure is fully compensated for. In this case the error is 0, and we call the compensation *complete*. In the development above we assume only one erasure, i.e. that none of the $\{a_k | k \in S_i\}$ are erased during the transmission.

Projection-based compensation can be generalized to the sequential erasure of multiple expansion coefficients, allowing a subset of the remaining coefficients for each compensation. The sets S_i of coefficients used to compensate each of the erasures are part of the system design constraints. We assume that once a coefficient has been erased and compensated for, it is not used to compensate for subsequent erasures. Under these assumptions the following can be shown:

(a) Superposition: The linearity of projections implies that the superposition of the optimal compensation of a_i and a_j using the same set of the remaining coefficients $S_i = S_j$, (i.e. $j \notin S_i$) produces the same error as the optimal compensation of the vector $a_i \mathbf{f}_i + a_j \mathbf{f}_j$ using the same set.

(b) Sequential superposition: If one of the coefficients $a_k, k \in S_i$ used in the compensation of a_i is subsequently erased and the remaining coefficients of S_i are used to optimally compensate for the erasure of the updated $\hat{a}_k$, this is equivalent to optimally compensating both a_i and a_k with the remaining coefficients in S_i.

(c) Sequential complete compensation: If an $a_k, k \in S_i$ used in the compensation of a_i is subsequently erased but completely compensated, the compensation of a_i is still optimal.

Projections can be used at the transmitter to intentionally introduce erasures before transmission, a process known as *puncturing*. Erasures compensated for with projections can be the basis for algorithms that sparsify dense representations. They can also be combined with quantization, in which the combined error is projected to the remaining coefficients, as described in [7, 8], although not necessarily in a data-independent ordering. This, however, is beyond the scope of this paper, and it is not discussed here.

4. TRANSMITTER-AWARE COMPENSATION

The projections are straightforward to implement if the transmitter is aware of the erasure occurrence. For the remaining of this paper we assume the coefficients are transmitted in sequence indexed by k in (2). We focus on causal compensation in which only a finite number of coefficients, subsequent to the erasure can be used for compensation. For clarity of the exposition, we assume a shift invariant frame

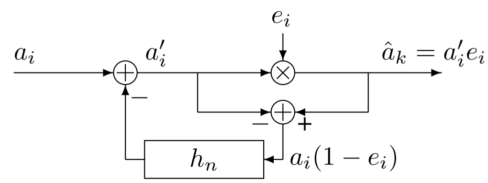

Fig. 1. Erasure-aware transmitter projecting erasure errors.

in the figures, although we develop the algorithms for arbitrary frames. A shift invariant frame has autocorrelation that is a function only of the index difference, i.e. satisfies $R_{i,j} = R_{i-j,0} \equiv R_{|i-j|}$. Thus, $c_{i,i+k} = c_{0,k} \equiv c_k$, and a transmitter aware of the erasure occurrence can be implemented using the system in figure 1, in which the feedback impulse response is equal to:

$$h_n = \sum_{k=1}^{p} c_k \delta_{n-k}. \tag{11}$$

In the figure, e_k denotes a sequence of 1 and 0, which multiplicatively implements the erasures. The resemblance of the system to Sigma-Delta noise shaping systems is not accidental, given that projection based compensation of errors in frame expansions was introduced in [7, 8], as an extension of Sigma-Delta noise shaping to arbitrary frames.

The compensation is optimal if the erasures are such that there is only one erasure within p coefficients, or if p is such that the erasure compensation is complete. Otherwise it is only a locally optimal strategy which minimizes the *incremental* error after an erasure has occurred, subject to the design constraints.

For arbitrary, shift varying frames, the feedback should use a time-varying system $h_{n,m}$ in which the coefficients satisfy (10) at the corresponding time point. The output y_k of this system should be

$$y_k = \sum_i h_{k,i} x_i, \tag{12}$$

in which $x_i = a_i(1 - e_i)$ is the input, and

$$h_{k,i} = \begin{cases} c_{i,k} & \text{for } 0 < k - i \leq p, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

5. PRE-COMPENSATION WITH CORRECTION

In many systems, and particularly in streaming applications, the transmitter is not aware of the erasure occurrence. In such situations it is possible to pre-project the error at the transmitter side, assuming an erasure will occur. If the erasure does not occur, the receiver undoes the compensation.

To implement this algorithm the transmitter at step i updates coefficients $a_{i+1}, \ldots, a_{i+p}$ to

$$a'_{i+k} = a_{i+k} + c_{i,i+k} a'_i, \tag{14}$$

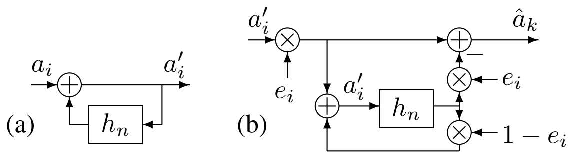

Fig. 2. (a) Transmitter and (b) receiver structure projecting erasure errors. Only the receiver is aware of the erasure.

where the $c_{i,i+k}$ satisfy (10). The a'_i used for the update is the coefficient as updated from all the previous iterations of the algorithm, not the original coefficient of the expansion, making the transmitter is a recursive system. Depending on the frame, this has the potential to generate instabilities, not addressed in this paper.

If an erasure does not occur the receiver at time step i receives coefficient a'_i. Otherwise it sets

$$a'_i = \sum_{k=1}^{p} c_{i-k,i} a'_{i-k}, \tag{15}$$

which is the part of a'_i from equation (14) that is due to the projection of the non-erased coefficients. Note that an erasure also erases the components of a'_i due to the projection of the previously received coefficients. Equation (15) ensures that these components are retained for the projection to be removed from the subsequently received coefficients, even though a'_i has not been received. The system outputs $\hat{a}_i$, depending on whether an erasure has occurred or not:

$$\hat{a}_i = (a'_i - \sum_{k=1}^{p} c_{i-k,i} a'_{i-k}) e_i \tag{16}$$

This removes the projection of the previously received coefficients from a'_i. It can be shown that this system is input-output equivalent to the system in section 4 [11]. To demonstrate this, it suffices to show that if a coefficient a'_i is not erased, then its projection is recursively removed from all the remaining non-erased coefficients. Any erased coefficient, has already been compensated for by the transmitter using (14). The reconstruction in equation (16) undoes the recursive effects of (14) and ensures that the projection only affects the p coefficients subsequent of the erasure. If the frame is shift invariant, the system looks like the one in figure 2, in which e_i,the sequence of ones and zeros denoting the erasures, is the same in all three locations in the figure. The system h_n is the same as in figure 1.

In several applications, such as packetized transmissions, frame expansions are used for transmission of blocks of coefficients. In such cases the systems described can be modified to accommodate block erasures by projecting the whole vector represented by the transmitted block to the subsequent coefficients.

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