

A Derivation of the Recursive Solution to the Autocorrelation Normal Equations

Norbert Wiener's formulation of extrapolation, interpolation, and smoothing of time series [8] naturally led to a set of linear equations referred to as autocorrelation normal equations (ACNE). In an appendix in [8], which was originally published as [4], Levinson presented an efficient recursive solution to ACNE later modified somewhat by Durbin [1]. This class of recursive solutions is typically referred to as the Levinson recursion or the Levinson–Durbin recursion. The structure and efficiency of this class of solutions derives from the basic structure of the associated matrix, specifically the fact that it is a symmetric Toeplitz matrix, together with the relation of the right-hand side of the matrix equation to the elements in the Toeplitz matrix. In this article, we derive a recursive solution to ACNE without reference to any of the motivating applications but exploiting the basic structure of the equations and straightforward properties of linear time-invariant (LTI) systems.

RELEVANCE

ACNE arise in a variety of applications including filter design, adaptive systems, signal prediction and estimation, time series analysis, data compression, system identification, and signal modeling. These applications are discussed in detail in [2]–[8]. They typically result from a problem formulation in these contexts associated with minimized mean square error. Efficient solution of ACNE forms the basis for both

real-time and offline implementation of many signal processing and data analysis systems.

PREREQUISITES

This article assumes a basic knowledge of discrete-time LTI systems including representation and analysis based on convolution and the z -transform. Also assumed is a basic familiarity with finite impulse response (FIR) and infinite impulse response (IIR) filters and the lattice filter structure. A general sense of how the autocorrelation normal equations might arise might be helpful for context but is not essential for understanding the development of the recursion.

PROBLEM STATEMENT

The focus of this article is on deriving a recursion for solving for the parameters $\alpha_k^{(p)}$ in ACNE, expressed in matrix form as

$$\begin{bmatrix} r_{ss}[0] & r_{ss}[1] & \cdots & r_{ss}[p-1] \\ r_{ss}[1] & r_{ss}[0] & \cdots & r_{ss}[p-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_{ss}[p-1] & r_{ss}[p-2] & \cdots & r_{ss}[0] \end{bmatrix} \times \begin{bmatrix} \alpha_1^{(p)} \\ \alpha_2^{(p)} \\ \vdots \\ \alpha_p^{(p)} \end{bmatrix} = \begin{bmatrix} r_{ss}[1] \\ r_{ss}[2] \\ \vdots \\ r_{ss}[p] \end{bmatrix} \quad (1)$$

The entries $r_{ss}[m]$ in (1) are symmetric in m , i.e.,

$$r_{ss}[m] = r_{ss}[-m]. \quad (2)$$

The matrix on the left-hand side of (1) is symmetric and Toeplitz. An important consequence of this is that the operation consisting of reversing the order of its columns, followed by reversing the

order of its rows, will result in the original matrix. This also implies that an equivalent equation to (1) may be obtained by flipping the two column vectors in this equation upside down and leaving the matrix unchanged. Doing this and using (2) allows (1) to be rewritten as

$$\begin{bmatrix} r_{ss}[0] & r_{ss}[1] & \cdots & r_{ss}[p-1] \\ r_{ss}[1] & r_{ss}[0] & \cdots & r_{ss}[p-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_{ss}[p-1] & r_{ss}[p-2] & \cdots & r_{ss}[0] \end{bmatrix} \times \begin{bmatrix} \alpha_p^{(p)} \\ \alpha_{p-1}^{(p)} \\ \vdots \\ \alpha_1^{(p)} \end{bmatrix} = \begin{bmatrix} r_{ss}[-p] \\ r_{ss}[-(p-1)] \\ \vdots \\ r_{ss}[-1] \end{bmatrix} \quad (3)$$

SOLUTION

Writing (1) and (3) in summation form and using (2), we respectively obtain

$$r_{ss}[i] - \sum_{m=1}^p \alpha_m^{(p)} r_{ss}[i-m] = 0, \quad i = 1, \dots, p \quad (4)$$

and

$$r_{ss}[i-p-1] - \sum_{m=1}^p \alpha_{p-m+1}^{(p)} r_{ss}[i-m] = 0, \quad i = 1, \dots, p. \quad (5)$$

We next observe that the conditions in (1) and (3), and respectively in (4) and (5), can be written in terms of convolution sums as either

$$f^{(p)}[n] = \sum_{m=-\infty}^{\infty} \alpha^{(p)}[m] r_{ss}[n-m] = 0, \quad n = 1, \dots, p \quad (6)$$

or

$$g^{(p)}[n] = \sum_{m=-\infty}^{\infty} c^{(p)}[m] r_{ss}[n-m] = 0, \quad n = 1, \dots, p. \quad (7)$$

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The sequences $a^{(p)}[n]$ and $c^{(p)}[n]$ in (6) and (7) are composed of the parameters $\alpha_k^{(p)}$ as

$$a^{(p)}[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ -\alpha_n^{(p)}, & 1 \leq n \leq p \\ 0, & n > p \end{cases} \quad (8)$$

and

$$c^{(p)}[n] = \begin{cases} 0, & n < 1 \\ -\alpha_{p-n+1}^{(p)}, & 1 \leq n \leq p \\ 1, & n = p + 1 \\ 0, & n > p + 1 \end{cases} \quad (9)$$

Denoting the z -transform of $a^{(p)}[n]$ as $A^{(p)}(z)$ and the z -transform of $c^{(p)}[n]$ as $C^{(p)}(z)$, we observe from (8) and (9) that $A^{(p)}(z)$ and $C^{(p)}(z)$ are related according to the following equation:

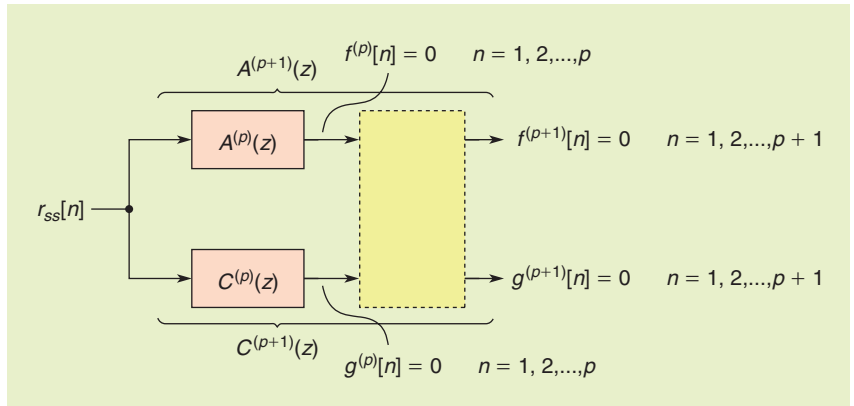
$$C^{(p)}(z) = z^{-(p+1)}A^{(p)}(1/z), \quad (10)$$

and equivalently,

$$A^{(p)}(z) = z^{-(p+1)}C^{(p)}(1/z). \quad (11)$$

Since $c^{(p)}[n]$ and $a^{(p)}[n]$ are related through (10) and equivalently (11), determining either the coefficients that form $a^{(p)}[n]$ in (6) or the coefficients that form $c^{(p)}[n]$ in (7) would in effect represent a solution to ACNE.

The general strategy in developing the recursion is to use the sequences $a^{(p)}[n]$ and $c^{(p)}[n]$, i.e., the p th-order solutions to (6) and (7), to compute the $(p+1)$ st-order solutions. The key observation in doing this is to view the sequences $a^{(p)}[n]$ and $c^{(p)}[n]$ as being the impulse responses of LTI systems respectively denoted $A^{(p)}(z)$ and $C^{(p)}(z)$, with (6) and (7) representing conditions on the respective output signals when both of the systems are driven by the input sequence $r_{ss}[n]$. With this in mind, obtaining a $(p+1)$ st-order solution to ACNE from a p th-order solution can be accomplished by combining the systems $A^{(p)}(z)$ and $C^{(p)}(z)$ to result in systems $A^{(p+1)}(z)$ and $C^{(p+1)}(z)$ whose impulse responses $a^{(p+1)}[n]$ and $c^{(p+1)}[n]$ satisfy the $(p+1)$ st-order conditions in (6) and (7).



[FIG1] A general approach behind deriving a step in the recursion.

The general approach in deriving a step in the recursion is depicted in Figure 1. Referring to this figure, there are two requirements of the system to be placed in the dashed box:

- 1) The signal $f^{(p+1)}[n]$ must be zero for $n = 1, \dots, p+1$, i.e., $A^{(p+1)}(z)$ must satisfy the $(p+1)$ st-order condition in (6).
- 2) The system $C^{(p+1)}(z)$ must be related to $A^{(p+1)}(z)$ as

$$C^{(p+1)}(z) = z^{-(p+2)}A^{(p+1)}(1/z), \quad (12)$$

i.e., (10) must be satisfied, resulting in a system $C^{(p+1)}(z)$ and related signal $g^{(p+1)}[n]$ that satisfy the $(p+1)$ st-order condition in (7).

In satisfying the first requirement, we use the fact that given two arbitrary scalars and an appropriately chosen scale factor for one of them, the two can be linearly combined to sum to zero. In particular, we satisfy the first requirement by constructing a system that obtains $f^{(p+1)}[n]$ by appropriately scaling the sequence $g^{(p)}[n]$ and subtracting it from $f^{(p)}[n]$, with the scale factor being chosen so that the $(p+1)$ st value of the resulting sequence is zero. The first through p th values will naturally remain zero as well, resulting in a sequence $f^{(p+1)}[n]$ that satisfies the first requirement. Written formally, the first requirement is satisfied by setting

$$f^{(p+1)}[n] = f^{(p)}[n] - k_{p+1}g^{(p)}[n], \quad (13)$$

with the scale factor k_{p+1} in (13) being computed as

$$k_{p+1} = \frac{f^{(p)}[p+1]}{g^{(p)}[p+1]}. \quad (14)$$

The system $A^{(p+1)}(z)$ is likewise related to $A^{(p)}(z)$ and $C^{(p)}(z)$ as

$$A^{(p+1)}(z) = A^{(p)}(z) - k_{p+1}C^{(p)}(z), \quad (15)$$

in effect expressing the relationships in (6), (7), and (13) using z -transforms as depicted in Figure 1.

The second requirement specifies that (12) must be satisfied. We now demonstrate that this will occur if $g^{(p+1)}[n]$ is computed as

$$g^{(p+1)}[n] = g^{(p)}[n-1] - k_{p+1}f^{(p)}[n-1] \quad (16)$$

when (10) has been satisfied in the previous iteration. In particular, (16) implies that $C^{(p+1)}(z)$ is related to $A^{(p)}(z)$ and $C^{(p)}(z)$ as

$$C^{(p+1)}(z) = z^{-1}C^{(p)}(z) - k_{p+1}z^{-1}A^{(p)}(z), \quad (17)$$

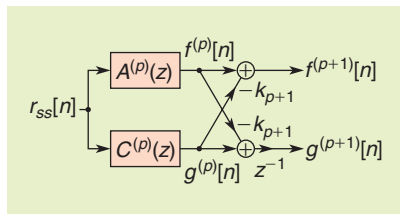
and substituting in the expression for $C^{(p)}(z)$ in (10) results in

$$C^{(p+1)}(z) = z^{-(p+2)}A^{(p)}(1/z) - k_{p+1}z^{-1}A^{(p)}(z). \quad (18)$$

Substituting (11) into (18), we obtain

$$C^{(p+1)}(z) = z^{-(p+2)}A^{(p)}(1/z) - k_{p+1}z^{-(p+2)}C^{(p)}(1/z), \quad (19)$$

which, invoking (15), simplifies to (12).



[FIG2] A system generating one step of the recursion.

The resulting system that generates one step of the recursion is depicted in Figure 2. Referring to this figure, replacing the input signal $r_{ss}[n]$ with $\delta[n]$ will produce a signal $a^{(p+1)}[n]$ at the output of the top branch from which the $(p + 1)$ st-order coefficients $-\alpha_k^{(p+1)}$ can be obtained using (8).

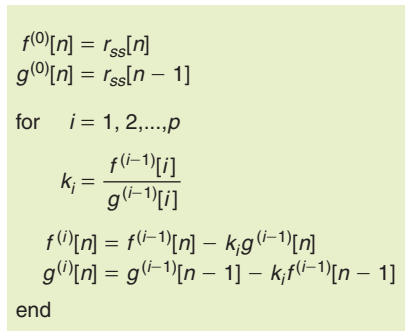
The final step in deriving the recursion involves selecting appropriate systems $A^{(0)}(z)$ and $C^{(0)}(z)$ for the base case. It is straightforward to verify that by using

$$A^{(0)}(z) = 1, \quad (20)$$

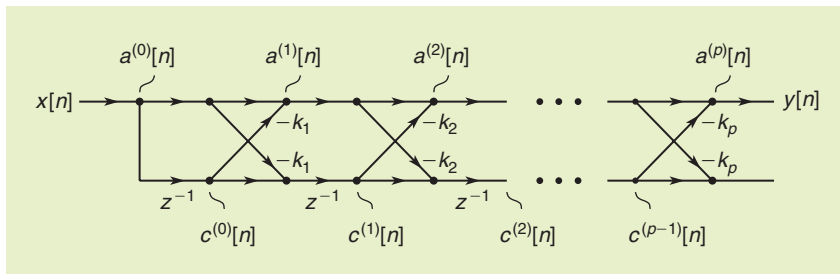
and with $C^{(0)}(z)$ being obtained from (10) as

$$C^{(0)}(z) = z^{-1}, \quad (21)$$

the first-order conditions in (6) and (7) will be satisfied after the initial iteration. Setting $r_{ss}[n]$ as the input, the signals $f^{(0)}[n]$ and $g^{(0)}[n]$ are in turn $f^{(0)}[n] = r_{ss}[n]$ and $g^{(0)}[n] = r_{ss}[n - 1]$, and we have all the pieces required to write a recursion for computing the coefficients k_i , which is listed in Figure 3. The resulting structure, depicted in Figure 4,



[FIG3] Recursion for creating a system whose impulse response contains the solution to the p th-order ACNE.

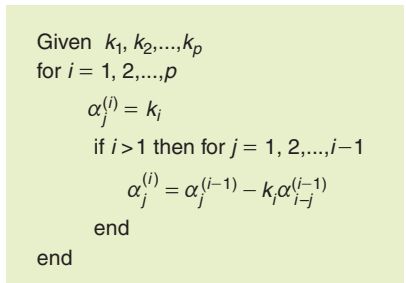


[FIG4] A structure obtained using the recursion in Figure 3.

may accordingly be excited by the unit impulse $\delta[n]$ to obtain a sequence whose elements contain the solution to ACNE as described in (8).

The FIR structure in Figure 4 is often referred to as the FIR lattice structure, and in both its FIR and IIR forms has become an important basic structure for implementing FIR and IIR difference equations because of its many desirable properties. In addition to it being a natural by-product of the development of the recursive solution to ACNE, it has also been a by-product of various specific applications such as acoustic tube modeling of the vocal tract, linear prediction and whitening filters. In these contexts, the lattice filter coefficients k_i have been referred to as reflection coefficients [5] or as the partial autocorrelation coefficients [2].

As discussed in [6, Sec. 6.6], the structure of the FIR lattice naturally leads to a recursion for obtaining the coefficients $\alpha_k^{(p)}$ from the coefficients k_i in Figure 4, and equivalently the sequence $a^{(p)}[n]$ through (8). That recursion is reproduced in Figure 5. The combination of Figures 3 and 5 then provide a recursive solution to ACNE and in effect correspond to or are at least equivalent to the Levinson recursion.



[FIG5] Algorithm for converting from k -parameters to FIR filter coefficients.

CONCLUSIONS

ACNE plays an important role in many aspects of signal processing and time series analysis. The recursive solution to these equations presented in this article is developed independently of any specific application context and utilizes the basic principles of LTI systems, resulting in the lattice filter structure as a by-product.

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