Proof: Any Φ is a 0-design, and Φ is a 1-design if and only if

$$\sum_{i=1}^{n} \phi_i = 0$$

which automatically holds if Φ is symmetric. By Corollary 3.2, Φ is a 2-design, and such a 0-, 1-, 2-design is a 3-design if Φ is symmetric. □

This result can be found in the literature (cf. [5] and [10]).

In Eldar and Forney [6], the relationship between tight frames and rank-one quantum measurements is investigated. It is shown that rank-one generalized quantum measurements (or positive operator-valued measures (POVMs)) correspond to tight frames.

It is hoped, that by drawing attention to the fact that WBE sequences, isometric (normalized, uniform) tight frames and 2-designs are the same thing, that the respective communities can benefit from each others' endeavours. Clearly, such an object, which has appeared independently in different areas is of interest, and deserves to be understood in this wider context.

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On the Asymptotic Performance of the Decorrelator

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***Abstract*—We derive the asymptotic signal-to-interference ratio (SIR) of the decorrelator in the large system limit, both for the case in which the number of users exceeds the spreading gain and for the case in which the number of users is less than the spreading gain. We show that, contrary to what is claimed in [1], [2], when the number of users exceeds the spreading gain and the decorrelator is defined in terms of the Moore–Penrose pseudoinverse, the SIR does *not* converge to zero.**

***Index Terms*—Code-division multiple access (CDMA), decorrelator, multiuser detection, signal-to-interference ratio (SIR), Wishart matrix.**

I. Introduction

In a code-division multiple-access (CDMA) system, each user transmits information by modulating a unique signature sequence. Often times, modeling the signature sequences as random can be appropriate [2]–[4]. For example, the signature sequences may be pseudonoise (PN) sequences that span many symbol periods, or the signature sequences may be effectively random due to independent multipath fading in the channel.

In recent studies, asymptotic expressions for the signal-to-interference ratio (SIR) of the decorrelator receiver [5] in the large system limit have been derived, assuming random signature sequences and power control. The large system limit implies that both the number of users M and the spreading gain N approach infinity with their ratio, $\beta = M/N$, held constant. In [1]–[4], [6] it was shown that for $\beta < 1$, the SIR for each user at the decorrelator output converges to $A^2(1-\beta)/\sigma^2$, where A^2 is the received power and σ^2 is the noise variance. The case $\beta > 1$ is not analyzed in [3], [4], [6]; in [1], [2] it is claimed incorrectly that in this case the SIR converges to 0, where the decorrelator is defined in terms of the Moore–Penrose pseudoinverse of the signature matrix.

In this correspondence, we derive the SIR for each user at the decorrelator output for both $\beta < 1$ and $\beta > 1$ in a unified manner. After reviewing the decorrelator in Section II, in Section III, we derive the asymptotic SIR of the decorrelator in the large system limit for all β. In particular, we show that the SIR does not converge to zero when $\beta > 1$.

The method of analysis we present is broadly applicable to other multiuser detectors as well. For example, it can be used as an alternative to methods in [1], [4] for deriving the asymptotic SIR for the matched-filter detector, and it has been recently used to derive the asymptotic SIR for the orthogonal multiuser detector [7] and the covariance shaping multiuser detector [8], [9].

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II. The Decorrelator

Consider an M-user white Gaussian synchronous CDMA system where each user transmits information by modulating a signature sequence. The discrete-time model for the received signal $\boldsymbol{r}$ is given by

$$\boldsymbol{r} = \boldsymbol{S}\boldsymbol{A}\boldsymbol{b} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{S} = [\boldsymbol{s}_1|\boldsymbol{s}_2|\cdots|\boldsymbol{s}_M]$ is the $N \times M$ matrix of signatures with $\boldsymbol{s}_m \in \mathbb{C}^N$ being the signature vector of the mth user, $\boldsymbol{A} = \operatorname{diag}(A_1, \ldots, A_M)$ is the matrix of received amplitudes with A_m being the amplitude of the mth user's signal, $\boldsymbol{b} = [b_1, b_2, \ldots, b_M]$ is the data vector with b_m being the mth user's transmitted symbol, and $\boldsymbol{n}$ is a noise vector whose elements are independent $\mathcal{CN}(0, \sigma^2)$. We assume that all data vectors are equally likely with covariance $\boldsymbol{I}$.

The decorrelator receiver [5] demodulates the information transmitted by each user by premultiplying the received vector by $\boldsymbol{S}^\dagger$, where $(\cdot)^\dagger$ denotes the *Moore–Penrose pseudoinverse* [10]. The matrix $\boldsymbol{S}^\dagger$ can be expressed directly in terms of the singular value decomposition (SVD) of $\boldsymbol{S}$ as follows. Let

$$\boldsymbol{S} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^* \tag{2}$$

where $\boldsymbol{U}$ is a unitary $N \times N$ matrix, $\boldsymbol{V}$ is a unitary $M \times M$ matrix, and $\boldsymbol{\Sigma}$ is a diagonal $N \times M$ matrix with diagonal elements $\sigma_i > 0$, $i = 1, \ldots, r$ and 0 otherwise, where r is the rank of $\boldsymbol{S}$. Then

$$\boldsymbol{S}^\dagger = \boldsymbol{V}\boldsymbol{\Sigma}^\dagger\boldsymbol{U}^* \tag{3}$$

where $\boldsymbol{\Sigma}^\dagger$ is a diagonal $M \times N$ matrix with diagonal elements $1/\sigma_i$, $i = 1, \ldots, r$ and 0 otherwise. We may also express $\boldsymbol{S}^\dagger$ as

$$\boldsymbol{S}^\dagger = (\boldsymbol{S}^*\boldsymbol{S})^\dagger\boldsymbol{S}^* \tag{4}$$

where

$$(\boldsymbol{S}^*\boldsymbol{S})^\dagger = \boldsymbol{V}\boldsymbol{\Lambda}^\dagger\boldsymbol{V}^*. \tag{5}$$

Here, $\boldsymbol{\Lambda}$ is a diagonal $M \times M$ matrix with diagonal elements $\lambda_i = \sigma_i^2$ and $\boldsymbol{\Lambda}^\dagger$ is a diagonal matrix with diagonal elements $1/\lambda_i$, $i = 1, \ldots, r$ and 0 otherwise. If the columns of $\boldsymbol{S}$ are linearly independent, then (4) reduces to $\boldsymbol{S}^\dagger = (\boldsymbol{S}^*\boldsymbol{S})^{-1}\boldsymbol{S}^*$.

The vector output $\boldsymbol{x}$ of the decorrelator is given by

$$\boldsymbol{x} = \boldsymbol{S}^\dagger\boldsymbol{r} = \boldsymbol{S}^\dagger\boldsymbol{S}\boldsymbol{A}\boldsymbol{b} + \boldsymbol{S}^\dagger\boldsymbol{n}. \tag{6}$$

Combining (2) with (3) we have that

$$\boldsymbol{S}^\dagger\boldsymbol{S} = \boldsymbol{V}\tilde{\boldsymbol{I}}\boldsymbol{V}^* = P_{\mathcal{V}} \tag{7}$$

where $\tilde{\boldsymbol{I}}$ is a diagonal $M \times M$ matrix whose first r diagonal elements are equal to 1 and whose remaining diagonal elements are all equal 0, and $P_{\mathcal{V}}$ denotes the orthogonal projection onto the range space $\mathcal{R}(\boldsymbol{S}^*)$ of $\boldsymbol{S}^*$, which we denote by $\mathcal{V}$. Note, that we also have $\mathcal{V} = \mathcal{N}(\boldsymbol{S})^\perp$, where $\mathcal{N}(\boldsymbol{S})$ denotes the null space of $\boldsymbol{S}$. Substituting (7) into (6)

$$\boldsymbol{x} = P_{\mathcal{V}}\boldsymbol{A}\boldsymbol{b} + \boldsymbol{S}^\dagger\boldsymbol{n}. \tag{8}$$

If the signature vectors $\{\boldsymbol{s}_m\}$ are linearly independent, then $r = M$ which implies that $\tilde{\boldsymbol{I}} = \boldsymbol{I}_M$, and from (7), $P_{\mathcal{V}} = \boldsymbol{V}\boldsymbol{V}^* = \boldsymbol{I}_M$. In this case, the data component in $\boldsymbol{x}$ is $\boldsymbol{A}\boldsymbol{b}$ so that all the multiple-access interference (MAI) is eliminated. However, if the signature vectors are linearly dependent, then $r < M$ and the off-diagonal elements of $P_{\mathcal{V}}$ are not all equal to 0, implying that there is MAI in the outputs.

To evaluate the SIR, we decompose each component x_m of the decorrelator output into

$$x_m = x_m^S + x_m^I + x_m^N \tag{9}$$

where the terms

$$x_m^S = [P_{\mathcal{V}}]_{mm}A_m b_m \tag{10}$$

$$x_m^I = \sum_{k \neq m} [P_{\mathcal{V}}]_{mk}A_k b_k \tag{11}$$

$$x_m^N = [(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_m^*\boldsymbol{S}^*\boldsymbol{n} \tag{12}$$

represent the desired signal, the MAI, and the noise, respectively. Here, $[\cdot]_{mk}$ and $[\cdot]_m$ denote, respectively, the mkth element and the mth column of the corresponding matrices, and we have used (4). From (10)–(12), the terms x_m^S, x_m^I, and x_m^N have variances

$$\operatorname{var}(x_m^S) = [P_{\mathcal{V}}]_{mm}^2 A_m^2 \tag{13}$$

$$\operatorname{var}(x_m^I) = [P_{\mathcal{V}}]_m^*\boldsymbol{A}^2[P_{\mathcal{V}}]_m - [P_{\mathcal{V}}]_{mm}^2 A_m^2 \tag{14}$$

$$\operatorname{var}(x_m^N) = \sigma^2[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm}. \tag{15}$$

The SIR at the mth output of the decorrelator is, therefore,

$$\gamma_m = \frac{[P_{\mathcal{V}}]_{mm}^2 A_m^2}{\sigma^2[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm} + [P_{\mathcal{V}}]_m^*\boldsymbol{A}^2[P_{\mathcal{V}}]_m - [P_{\mathcal{V}}]_{mm}^2 A_m^2}. \tag{16}$$

III. Asymptotic Large-System Performance

We now evaluate the SIR in the large-system limit when accurate power control and random Gaussian signatures are used. In the case of accurate power control, i.e., $\boldsymbol{A} = A\boldsymbol{I}_M$, we can simplify (16) to

$$\gamma_m = \frac{[P_{\mathcal{V}}]_{mm}^2}{\zeta[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm} + [P_{\mathcal{V}}]_{mm} - [P_{\mathcal{V}}]_{mm}^2} \tag{17}$$

where

$$\frac{1}{\zeta} = \frac{A^2}{\sigma^2} \tag{18}$$

is the received signal-to-noise ratio (SNR). Next, we consider the limit $M \to \infty$ with $\beta \triangleq M/N$ held constant when the elements of the $N \times M$ signature matrix $\boldsymbol{S}$ are independent $\mathcal{CN}(0, 1/N)$. The following theorem characterizes the SIR of the decorrelator in this limit.

Theorem 1: Let the elements of the $N \times M$ signature matrix $\boldsymbol{S}$ be independent $\mathcal{CN}(0, 1/N)$, and let the matrix of amplitudes $\boldsymbol{A}$ be expressible as $A\boldsymbol{I}_M$. Then, in the limit as $M \to \infty$ with $\beta \triangleq M/N$ held constant, the SIR for each user at the decorrelator output satisfies[1]

$$\gamma_m \xrightarrow{\text{m.s.}} \begin{cases} \frac{1-\beta}{\zeta}, & \beta \leq 1 \\ \frac{\beta-1}{(\beta-1)^2+\zeta\beta}, & \beta > 1. \end{cases} \tag{19}$$

Proof: The proof of Theorem 1 relies heavily on the following lemma involving Wishart matrices, which have the form $\boldsymbol{W} = \boldsymbol{S}^*\boldsymbol{S}$ where the elements of $\boldsymbol{S}$ are independent $\mathcal{CN}(0, \sigma^2)$. Although this lemma can be found in the statistical literature (see, e.g., [12]), a direct and straightforward proof is given in Appendix B. The lemma and its proof rely on the concepts of isotropically distributed vectors and matrices, which are reviewed in Appendix A.

[1]We use the notation $\xrightarrow{\text{m.s.}}$ to denote convergence in the mean-squared ($\boldsymbol{L}^2$) sense [11].

Lemma 1: Let the elements of an $N \times M$ matrix $\boldsymbol{S}$ be independent $\mathcal{CN}(0, \sigma^2)$. Then the eigenvector matrix of $\boldsymbol{W} = \boldsymbol{S}^*\boldsymbol{S}$ is isotropically distributed unitary and independent of the eigenvalues.

To prove (19) we need to determine the limits of $[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm}$ and $[P_{\mathcal{V}}]_{mm}$.

From (5), the quantity $[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm}$ can be written as

$$[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm} = [\boldsymbol{V}\boldsymbol{\Lambda}^\dagger\boldsymbol{V}^*]_{mm} = \boldsymbol{v}_m^*\boldsymbol{\Lambda}^\dagger\boldsymbol{v}_m \tag{20}$$

where $\boldsymbol{V}$ and $\boldsymbol{\Lambda}$ are the eigenvector and eigenvalue matrices in the eigendecomposition of the Wishart matrix $\boldsymbol{S}^*\boldsymbol{S}$ and $\boldsymbol{v}_m$ is the mth column of $\boldsymbol{V}^*$. Thus, using Lemma 1, we conclude that $\boldsymbol{V}^*$ is an isotropically distributed unitary matrix independent of $\boldsymbol{\Lambda}$. Since $\boldsymbol{v}_m$ is a column of an isotropically distributed unitary matrix, from Appendix A it follows that $\boldsymbol{v}_m$ is an isotropically distributed unit vector. Consequently, $\boldsymbol{v}_m$ has the same distribution as $\boldsymbol{z}/\sqrt{\boldsymbol{z}^*\boldsymbol{z}}$, where $\boldsymbol{z}$ is an M-dimensional vector of independent $\mathcal{CN}(0, 1)$ random variables. Thus, $[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm}$ has the same distribution as

$$\frac{\boldsymbol{z}^*\boldsymbol{\Lambda}^\dagger\boldsymbol{z}}{\boldsymbol{z}^*\boldsymbol{z}} = \frac{\sum_{j=1}^{M} \alpha_j |z_j|^2/M}{\sum_{j=1}^{M} |z_j|^2/M} \tag{21}$$

where z_j is the jth component of $\boldsymbol{z}$, and

$$\alpha_j = \begin{cases} \frac{1}{\lambda_j}, & \lambda_j \neq 0 \\ 0, & \lambda_j = 0 \end{cases} \tag{22}$$

with λ_j denoting the jth eigenvalue of $\boldsymbol{S}^*\boldsymbol{S}$. To evaluate the limit of (21) when $M \to \infty$, we rely on the following pair of lemmas.

Lemma 2 [13]: If the ratio of the number of users to the signature length is, or converges to, a constant

$$\lim_{M\to\infty} \frac{M}{N} = \beta \in (0, \infty) \tag{23}$$

then the percentage of the M eigenvalues of $\boldsymbol{S}^*\boldsymbol{S}$ that lie below x converges to the cumulative distribution function of the probability density function

$$f_\beta(x) = [1 - \beta^{-1}]^+ \delta(x) + \frac{\sqrt{[x-\eta_1]^+[\eta_2 - x]^+}}{2\pi\beta x} \tag{24}$$

where

$$\eta_1 = \left(1 - \sqrt{\beta}\right)^2 \tag{25}$$

$$\eta_2 = \left(1 + \sqrt{\beta}\right)^2 \tag{26}$$

and the operator $[\cdot]^+$ is defined according to

$$[u]^+ \triangleq \max\{0, u\}. \tag{27}$$

Lemma 3: Let $\{c_j\}$ denote a set of independent and identically distributed (i.i.d.) random variables independent of $\{\lambda_j\}$ with $E((c_1 - E(c_1))^2) < \infty$, where $\{\lambda_j\}$ denote the eigenvalues of a Wishart matrix under the conditions of Lemma 2. Let $g(\cdot)$ be a function such that $g(0) < \infty$ and $g(x) < \infty$ for $x \in [(1-\sqrt{\beta})^2, (1+\sqrt{\beta})^2]$. Then, as $M \to \infty$

$$\frac{1}{M}\sum_{j=1}^{M} g(\lambda_j)c_j \xrightarrow{\text{m.s.}} E(g(\lambda_1))E(c_1) \tag{28}$$

where $E(g(\lambda_1))$ is evaluated according to the probability density function $f_\beta(x)$ of (24).

Proof: Let

$$\kappa_M = \frac{1}{M}\sum_{j=1}^{M} g(\lambda_j)\tilde{c}_j \tag{29}$$

where $\tilde{c}_j = c_j - E(c_j)$. Then

$$E(\kappa_M) = \frac{1}{M}\sum_{j=1}^{M} E(g(\lambda_j))E(\tilde{c}_j) = 0 \tag{30}$$

and

$$\begin{aligned} E(\kappa_M^2) &= \frac{1}{M^2}\sum_{j=1}^{M}\sum_{k=1}^{M} E(g(\lambda_j)g(\lambda_k))E(\tilde{c}_j\tilde{c}_k) \\ &= \frac{1}{M^2}\sum_{j=1}^{M} E(g^2(\lambda_j))E(\tilde{c}_j^2) \\ &= \frac{1}{M} E(g^2(\lambda_1))E(\tilde{c}_1^2) \end{aligned} \tag{31}$$

where we have used the fact that the λ_j's are identically distributed, as are the $\tilde{c}_j$'s. Since by assumption $E(\tilde{c}_1^2) < \infty$, it follows from (32) that if $E(g^2(\lambda_1))$ is bounded as $M \to \infty$, then $E(\kappa_M^2) \to 0$ as $M \to \infty$. In [13], it is shown that for $\beta \neq 1$, the smallest nonzero eigenvalue converges almost surely to $\lambda_{\min} = (1-\sqrt{\beta})^2$ and the largest eigenvalue converges almost surely to $\lambda_{\max} = (1+\sqrt{\beta})^2$. Therefore, $E(g^2(\lambda_1))$ is bounded as $M \to \infty$ as long as $g(0) < \infty$ and $g(x)$ is bounded on $[(1-\sqrt{\beta})^2, (1+\sqrt{\beta})^2]$, which is satisfied by the assumptions of the lemma. We conclude that $\lim_{M\to\infty} E(\kappa_M^2) = 0$ so that

$$\lim_{M\to\infty} \kappa_M = E(\kappa_M) = 0 \tag{32}$$

where the limit is to be understood in a mean-squared sense.

Combining (32) with (29)

$$\lim_{M\to\infty} \frac{1}{M}\sum_{j=1}^{M} g(\lambda_j)c_j = \lim_{M\to\infty} \frac{E(c_1)}{M}\sum_{j=1}^{M} g(\lambda_j) \tag{33}$$

where the limits are to be understood in a mean-squared sense. From Lemma 2

$$\begin{aligned} \lim_{M\to\infty} \frac{E(c_1)}{M}\sum_{j=1}^{M} g(\lambda_j) &= E(c_1)\int_0^\infty g(x)f_\beta(x)\,dx \\ &= E(c_1)E(g(\lambda_1)) \end{aligned} \tag{34}$$

where we used the fact that for $M \to \infty$ the nonzero eigenvalues λ_j are all in the interval $[(1-\sqrt{\beta})^2, (1+\sqrt{\beta})^2]$ [13], and the assumptions of the lemma that $g(0) < \infty$ and $g(x)$ is bounded on $[(1-\sqrt{\beta})^2, (1+\sqrt{\beta})^2]$. Combining (34) with (33) completes the proof of the lemma.□

Applying Lemma 3 and the strong law of large numbers to the numerator and denominator of (21), respectively, and noting that in this case $g(x) = 0$ for $x = 0$ and $g(x) = 1/x$ for $x > 0$ so that $g(x)$ satisfies the conditions of Lemma 3, we have

$$[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm} \xrightarrow{\text{m.s.}} \frac{E(\alpha_1)E(|z_1|^2)}{E(|z_1|^2)} = E(\alpha_1) \tag{35}$$

as $M \to \infty$, where $E(\alpha_1)$ is evaluated according to the probability density function $f_\beta(x)$ of (24).

Similarly, from (7), $[P_{\mathcal{V}}]_{mm}$ can be written as

$$[P_{\mathcal{V}}]_{mm} = [\boldsymbol{V}\tilde{\boldsymbol{I}}\boldsymbol{V}^*]_{mm} = \boldsymbol{v}_m^* \tilde{\boldsymbol{I}} \boldsymbol{v}_m \tag{36}$$

which has the same distribution as

$$\frac{\boldsymbol{z}^* \tilde{\boldsymbol{I}} \boldsymbol{z}}{\boldsymbol{z}^* \boldsymbol{z}} = \frac{\sum_{j=1}^{M} \mu_j |z_j|^2 / M}{\sum_{j=1}^{M} |z_j|^2 / M} \tag{37}$$

where

$$\mu_j = \begin{cases} 1, & \lambda_j \neq 0 \\ 0, & \lambda_j = 0. \end{cases} \tag{38}$$

Applying Lemma 3 and the strong law of large numbers to the numerator and denominator of (37), respectively, and noting that in this case $g(x) = 0$ for $x = 0$ and $g(x) = 1$ for $x > 0$ so that $g(x)$ satisfies the conditions of Lemma 3, we have

$$[P_{\mathcal{V}}]_{mm} \xrightarrow{\text{m.s.}} \frac{E(\mu_1)E(|z_1|^2)}{E(|z_1|^2)} = E(\mu_1) \tag{39}$$

as $M \to \infty$, where $E(\mu_1)$ is evaluated according to the probability density function $f_\beta(x)$ of (24).

We now proceed to compute $E(\alpha_1)$ and $E(\mu_1)$. From (22) we have that

$$E(\alpha_1) = \lim_{M\to\infty} P(\lambda_1 \neq 0) E(1/\lambda_1 | \lambda_1 \neq 0). \tag{40}$$

Using (24)

$$\lim_{M\to\infty} P(\lambda_1 \neq 0) = \begin{cases} 1, & \beta \leq 1 \\ \frac{1}{\beta}, & \beta > 1. \end{cases} \tag{41}$$

Also, for $\beta \leq 1$

$$\begin{aligned} \lim_{M\to\infty} E(1/\lambda_1 | \lambda_1 \neq 0) &= \int_0^\infty \frac{\sqrt{[x-\eta_1]^+[\eta_2 - x]^+}}{2\pi\beta x^2}\,dx \\ &= \int_{\eta_1}^{\eta_2} \frac{\sqrt{(x-\eta_1)(\eta_2-x)}}{2\pi\beta x^2}\,dx \\ &= \frac{1}{1-\beta} \end{aligned} \tag{42}$$

and for $\beta > 1$

$$\begin{aligned} \lim_{M\to\infty} E(1/\lambda_1 | \lambda_1 \neq 0) &= \beta \int_0^\infty \frac{\sqrt{[x-\eta_1]^+[\eta_2 - x]^+}}{2\pi\beta x^2}\,dx \\ &= \beta \int_{\eta_1}^{\eta_2} \frac{\sqrt{(x-\eta_1)(\eta_2-x)}}{2\pi\beta x^2}\,dx \\ &= \frac{1}{\beta-1} \end{aligned} \tag{43}$$

where the integrals are evaluated using [14]. Thus,

$$[(\boldsymbol{S}^*\boldsymbol{S})^\dagger]_{mm} \xrightarrow{\text{m.s.}} E(\alpha_1) = \begin{cases} \frac{1}{1-\beta}, & \beta \leq 1 \\ \frac{1}{\beta(\beta-1)}, & \beta > 1. \end{cases} \tag{44}$$

Similarly

$$[P_{\mathcal{V}}]_{mm} \xrightarrow{\text{m.s.}} E(\mu_1) = \lim_{M\to\infty} P(\lambda_1 \neq 0) = \begin{cases} 1, & \beta \leq 1 \\ \frac{1}{\beta}, & \beta > 1. \end{cases} \tag{45}$$

It is well known that if $x_n \xrightarrow{\text{m.s.}} \overline{x}$ and $y_n \xrightarrow{\text{m.s.}} \overline{y}$, then $x_n \pm y_n \xrightarrow{\text{m.s.}} \overline{x} \pm \overline{y}$ and $x_n y_n \xrightarrow{\text{m.s.}} \overline{xy}$ [11]. The following lemma which involves the convergence of $1/x_n$ is now required to complete the proof of Theorem 1.

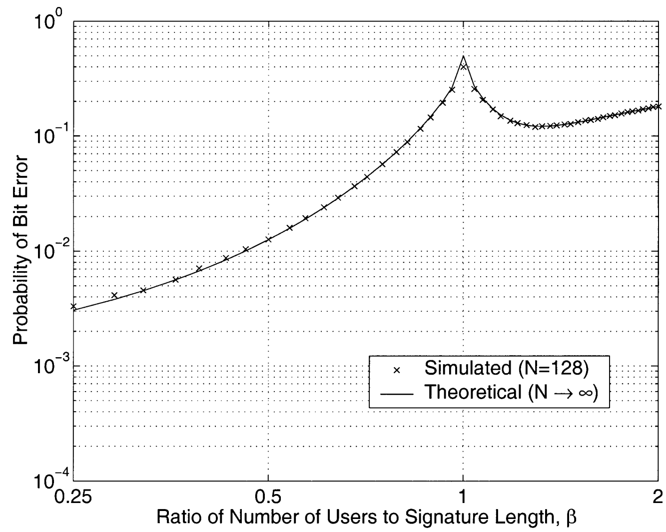

Fig. 1. Theoretical $(N \to \infty)$ and experimentally observed $(N = 128)$ performance of the decorrelator as a function of $\beta = M/N$, with equal-power users, random signatures, QPSK symbols, and an SNR per bit of 7 dB.

Lemma 4: Let $x_n \xrightarrow{\text{m.s.}} \overline{x}$, where $\{x_n\}$ is a sequence of random variables such that $|1/x_n| \leq B$ for all n, and $\overline{x} \neq 0$. Then

$$\frac{1}{x_n} \xrightarrow{\text{m.s.}} \frac{1}{\overline{x}}. \tag{46}$$

Proof:

$$\begin{aligned} E\left(\left(\frac{1}{x_n} - \frac{1}{\overline{x}}\right)^2\right) &= E\left(\left(\frac{\overline{x} - x_n}{x_n \overline{x}}\right)^2\right) \\ &\leq E\left(\left(\frac{B(\overline{x} - x_n)}{\overline{x}}\right)^2\right) \\ &= \frac{B^2}{\overline{x}^2} E\left((\overline{x} - x_n)^2\right) \\ &\to 0 \end{aligned} \tag{47}$$

since $x_n \xrightarrow{\text{m.s.}} \overline{x}$. □

Substituting (44) and (45) into (17), and using the fact that $\gamma_m \leq 1/\zeta$ with Lemma 4 completes the proof of Theorem 1. □

In Fig. 1, we plot the theoretical asymptotic large-system bit-error rate $(N \to \infty)$ of the decorrelator as a function of $\beta = M/N$ for quadrature phase-shift keying (QPSK) symbols, given by substituting (19) into the formula for the QPSK bit-error rate of a symbol-by-symbol threshold detector for additive white Gaussian noise (AWGN) channels [15]

$$P_e = \mathcal{Q}\left(\sqrt{\gamma_m}\right) \tag{48}$$

where

$$\mathcal{Q}(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-t^2/2}\,dt. \tag{49}$$

The theoretical bit-error rate is remarkably consistent with the simulated bit-error rate $(N = 128)$ of the decorrelator for QPSK symbols.

It is evident from Fig. 1 that the SIR of the decorrelator does not converge to zero in the large-system limit for $\beta > 1$, since the probability of error clearly does not converge to $1/2$.

Appendix A
Isotropically Distributed Vectors and Matrices

In this appendix, we define the concept of isotropically distributed vectors and matrices and highlight the key properties that are used to prove Lemma 1 and Theorem 1. A more detailed discussion can be found in [16].

Definition 1: An m-dimensional complex random vector $\boldsymbol{\phi}$ is isotropically distributed if its probability density is invariant to all unitary transformations; i.e., $f(\boldsymbol{\phi}) = f(\boldsymbol{\Theta}^* \boldsymbol{\phi})$ for all $\boldsymbol{\Theta}$ such that $\boldsymbol{\Theta}^*\boldsymbol{\Theta} = \boldsymbol{I}_m$.

Intuitively, an isotropically distributed complex vector is equally likely to point in any direction in complex space. Thus, the probability density of $\boldsymbol{\phi}$ is a function of its magnitude but not its direction. If, in addition, $\boldsymbol{\phi}$ is constrained to be a unit vector, then the probability density is

$$f(\boldsymbol{\phi}) = \frac{\Gamma(m)}{\pi^m} \delta(\boldsymbol{\phi}^*\boldsymbol{\phi} - 1) \tag{50}$$

and $\boldsymbol{\phi}$ is conveniently generated by $\boldsymbol{\phi} = \boldsymbol{z}/\sqrt{\boldsymbol{z}^*\boldsymbol{z}}$, where $\boldsymbol{z}$ is an m-dimensional vector of independent $\mathcal{CN}(0, 1)$ random variables.

Definition 2: An $n \times m$ complex random matrix $\boldsymbol{\Phi}$ is isotropically distributed if its probability density is unchanged when premultiplied by an $n \times n$ unitary matrix; i.e., $f(\boldsymbol{\Phi}) = f(\boldsymbol{\Theta}^*\boldsymbol{\Phi})$ for all $\boldsymbol{\Theta}$ such that $\boldsymbol{\Theta}^*\boldsymbol{\Theta} = \boldsymbol{I}_n$.

From the definition of an isotropically distributed matrix, it can be shown that the probability density is also unchanged when the matrix is postmultiplied by an $m \times m$ unitary matrix; i.e., $f(\boldsymbol{\Phi}) = f(\boldsymbol{\Phi}\boldsymbol{\Theta})$ for all $\boldsymbol{\Theta}$ such that $\boldsymbol{\Theta}^*\boldsymbol{\Theta} = \boldsymbol{I}_m$. Furthermore, by combining Definitions 1 and 2, we can readily see that the column vectors of $\boldsymbol{\Phi}$ are themselves isotropically distributed vectors.

Appendix B
Proof of Lemma 1

Let $\boldsymbol{S} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$ be the SVD [10] of $\boldsymbol{S}$, where $\boldsymbol{U}$ is an $N \times N$ unitary matrix, $\boldsymbol{V}$ is an $M \times M$ unitary matrix, and $\boldsymbol{\Sigma}$ is a diagonal $N \times M$ matrix with diagonal elements $\sigma_i \geq 0$. Then

$$\boldsymbol{W} = \boldsymbol{S}^*\boldsymbol{S} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^* \tag{51}$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^*\boldsymbol{\Sigma}$ is a diagonal matrix of eigenvalues of $\boldsymbol{W}$, and $\boldsymbol{V}$ is a matrix of eigenvectors of $\boldsymbol{W}$.

Let $\boldsymbol{\Upsilon}$ denote an independent and isotropically distributed unitary matrix. By premultiplying and postmultiplying (51) by $\boldsymbol{\Upsilon}^*$ and $\boldsymbol{\Upsilon}$, respectively, we have that

$$\boldsymbol{\Upsilon}^*\boldsymbol{S}^*\boldsymbol{S}\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^*\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^*\boldsymbol{\Upsilon} \tag{52}$$

or equivalently

$$(\boldsymbol{S}\boldsymbol{\Upsilon})^*(\boldsymbol{S}\boldsymbol{\Upsilon}) = (\boldsymbol{V}^*\boldsymbol{\Upsilon})^*\Lambda(\boldsymbol{V}^*\boldsymbol{\Upsilon}). \tag{53}$$

Let us examine the left-hand side of (53). Since the elements of $\boldsymbol{S}$ are $\mathcal{CN}(0, \sigma^2)$, $\boldsymbol{S}$ is an isotropically distributed matrix. With $\boldsymbol{S}$ being isotropically distributed and $\boldsymbol{\Upsilon}$ being unitary, $\boldsymbol{S}\boldsymbol{\Upsilon}$ has the same distribution as $\boldsymbol{S}$, and, consequently, $(\boldsymbol{S}\boldsymbol{\Upsilon})^*(\boldsymbol{S}\boldsymbol{\Upsilon})$ has the same distribution as $\boldsymbol{W} = \boldsymbol{S}^*\boldsymbol{S}$.

We now focus on the right-hand side of (53). Note that $\boldsymbol{V}^*\boldsymbol{\Upsilon}$ is unitary and $\boldsymbol{\Lambda}$ is diagonal, so that the right-hand side of (53) is an eigendecomposition. Now, since $\boldsymbol{\Upsilon}$ is an isotropically distributed unitary matrix and $\boldsymbol{V}^*$ is a unitary matrix, the eigenvector matrix $\boldsymbol{V}^*\boldsymbol{\Upsilon}$ is an isotropically distributed unitary matrix. Furthermore, the eigenvector matrix $\boldsymbol{V}^*\boldsymbol{\Upsilon}$ is independent of the eigenvalue matrix $\boldsymbol{\Lambda}$ because $\boldsymbol{\Upsilon}$ is independent of $\boldsymbol{\Lambda}$.

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