AN ITERATIVE RECONSTRUCTION ALGORITHM FOR AMPLITUDE SAMPLING

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ABSTRACT

Traditional data acquisition systems rely on sampling of bandlimited signals at uniformly spaced time instants with high-precision amplitude representation. Amplitude sampling, as previously introduced by the authors, is based on uniform amplitude sampling with high precision timing information. This paper presents an iterative algortihm for signal recovery from amplitude samples and compares the results with the use of iterative recovery from the nonuniform time samples implied by the amplitude sampling process. The iterative amplitude sampling reconstruction algorithm (IASR) proposed and evaluated in this paper is based on the interpretation of amplitude sampling as uniform sampling of an associated function obtained from a reversible transformation of the bandlimited input signal. The IASR algorithm exploits uniform sampling and the frequency- and time-domain properties of the amplitude-time function. The IASR algorithm is compared with the use of iterative nonuniform sampling reconstruction as proposed by Voroni. While in numerical simulations the Voronoi algorithm converges to marginally better reconstruction than IASR, the convergence rate of IASR is significantly better.

Index Terms— Nonuniform sampling and reconstruction, iterative algorithms, level-crossing sampling.

1. INTRODUCTION

Conventional digital data acquisition systems typically rely on uniform time sampling with no or very fine amplitude quantization. An alternative approach for signal acquisition is based on quantizing amplitude and representing time with no or very finely quantized values. One of the early instances of a theoretical basis for signal representation through amplitude quantization is found in Logan's theorem [1] in which a certain class of bandpass signals can be represented—up to a multiplicative constant—by only its zero crossings. An extension to multiple levels with quantized time was presented in [2]. The continuous-time version was later introduced in [3].

Logan's theorem does not provide a procedure for recovery from zero crossings. Reconstruction, although unstable, was later suggested by further imposing periodicity within the prescribed subset of bandpass signals [4]. Zero-crossing representation within the framework of wavelet transforms has also been studied in [5]. In this work, the recovery algorithm exploits additional information in order to stabilize the reconstruction. Level-crossing sampling processes are typically motivated by low-power consumption, thus prioritizing simplicity in the implementation [6]. In practice, reconstruction is commonly achieved by a zero- or first-order hold [7, 8].

In amplitude sampling as proposed in [9, 10], signal amplitude is quantized and time is represented with infinite precision. Conceptually, amplitude sampling can be viewed as a transformation of the source signal into a monotonic function. Samples are then taken whenever this resulting function crosses a predefined set of amplitude levels. An equivalent and more practical implementation is presented in Section 2 and Fig. 1. The iterative algorithm for signal reconstruction from the resulting time samples exploits the inherent structure of the amplitude-sampled signals and achieves faster convergence, in terms of number of iterations, than the Voronoi algorithm for nonuniform time sampling reconstruction.

2. BACKGROUND

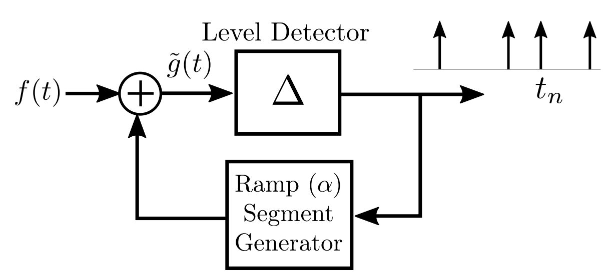

Fig. 1. Block-diagram representation.

In this section, we present the main results of amplitude

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sampling for signal representation proposed and derived in [9, 10]. As discussed in [9, 10], the conceptual idea is to add a ramp to the source signal with sufficient slope so that the resulting function is monotonic and the samples taken from the uniform amplitude crossing of the function can completely represent the source signal. Although the analysis and representation of the algorithm are presented in terms of this conceptual view of ramp addition, a more practical and completely equivalent implementation is shown in Fig. 1. In this implementation, the output impulses occur at times at which the input signal to the level detector takes the value $\Delta > 0$. Then, this sequence of time instants is used to generate ramp segments in $(t_n, t_{n+1}]$ with slope α that differ by an amplitude shift of $-\Delta$ between successive intervals. The slope α is chosen sufficiently large. Note that every function involved in this process is bounded. The sequence of time instants is explicitly related to the nonuniform time samples of the source signal, i.e. $f(t_n) = n\Delta - \alpha t_n$. Analytically, the sequence of time instants can be equivalently obtained by first adding a ramp with slope α to the source signal in order to generate a monotonic function, i.e. $g(t) = \alpha t + f(t)$. The output samples correspond to the level crossings of this function at amplitude levels $\{n\Delta\}$.

We denote the inverse of $u = g(t)$ by $\hat{g}(u)$, which can be expressed as $\hat{g}(u) = u/\alpha + h(u)$ for some function $h(u)$. In this domain, the relationship to the sequence of time instants is $t_n = \hat{g}(n\Delta)$, or equivalently

$$h(n\Delta) = t_n - n\Delta/\alpha \tag{1}$$

for all $n \in \mathbb{Z}$. The function $h(u)$ is then sampled uniformly, thus we can interpret the sampling process as uniform sampling of the amplitude-time function $h(u)$.

We denote by M_α the mapping from $f(t)$ to $h(u)$ with an appropriate choice of $\alpha \in \mathbb{R}$. Essentially, the function $h = M_\alpha f$ is obtained by adding a ramp of slope α to f, taking the inverse of the resulting function, and then subtracting a ramp of slope $1/\alpha$. It also follows in a straightforward manner that $M_{1/\alpha} h = f$, which implies a duality of properties between $f(t)$ and $h(u)$. For example, if $f(t)$ is bandlimited, then $h(u)$ is necessarily nonbandlimited. Through the duality of $f(t)$ and $h(u)$, if $h(u)$ is bandlimited, then $f(t)$ cannot be bandlimited. More specifically, if $f(t)$ is a function bandlimited to σ rad/s satisfying $|f(t)| \leq A$ and with appropriate decay conditions on the real line, then $h(u)$ is a nonbandlimited function whose Fourier transform additionally satisfies $H(\omega) = O(e^{-|\omega|a})$ as $\omega \to \infty$ for ω in rad/s and

$$a = \frac{|\alpha|}{\sigma} \log\left(\frac{|\alpha|}{A\sigma}\right) - \frac{|\alpha| - A\sigma}{\sigma}. \tag{2}$$

It should be emphasized that the decay depends on the logarithmic and linear difference between α and $A\sigma$. By duality, the same spectral properties hold for f when h is considered bandlimited.

3. ITERATIVE AMPLITUDE SAMPLING RECONSTRUCTION (IASR)

In this section, we present an iterative algorithm for amplitude sampling that achieves numerically accurate reconstruction. Assume that the source signal f is a real-valued function bandlimited to σ rad/s and bounded by some $A > 0$. By Bernstein's inequality [11], it follows that $\sup_{t\in\mathbb{R}} |f'(t)| \leq A\sigma$. It is then sufficient to choose $|\alpha| > A\sigma$ in order for $\alpha t + f(t)$ to be strictly monotonic—hence invertible. Without loss of generality, we will assume that the slope of the ramp is positive and satisfies $\alpha > A\sigma$ for ease of illustration.

3.1. The IASR algorithm

Amplitude sampling can be viewed as uniform sampling of h as indicated in (1). The approximate reconstruction shown in Fig. 2 is referred to as the bandlimited interpolation approximation (BIA) and was proposed in [9]. The BIA algorithm first performs a bandlimited interpolation of h parametrized by Δ and denoted by D/C in Fig. 2 . Its output is given by

$$h_\Delta(u) = \sum_{k\in\mathbb{Z}} h(k\Delta)\mathrm{sinc}(u/\Delta - k). \tag{3}$$

This approach exploits the exponential decay of the function h since we have that

$$||h - h_\Delta||_\infty \leq \frac{C'}{a} e^{-a\pi/\Delta} \tag{4}$$

for some $C' > 0$. There is no anti-aliasing filter preceding the sampling process of h. The aliasing error in the reconstruction is characterized in terms of the L^∞ error given by (4). This error can be reduced by increasing the difference $\alpha - A\sigma$ or by decreasing the level separation Δ. These two effects will later become important in the interpretation of the simulation results.

Next, h_Δ is processed by $M_{1/\alpha}$ to generate $f_\Delta = M_{1/\alpha} h_\Delta$, which assumes that the function $h_\Delta(u) + u/\alpha$ is invertible. This can always be achieved, in principle, through a combination of a sufficiently large α and/or a sufficiently small Δ.

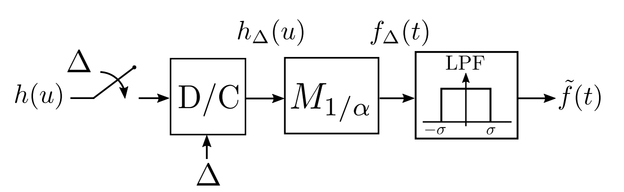


Fig. 2. Approximate reconstruction procedure for a bandlimited source signal f such that $h = M_\alpha f$. The block D/C is a discrete-to-continuous operation involving sinc interpolation with period Δ.

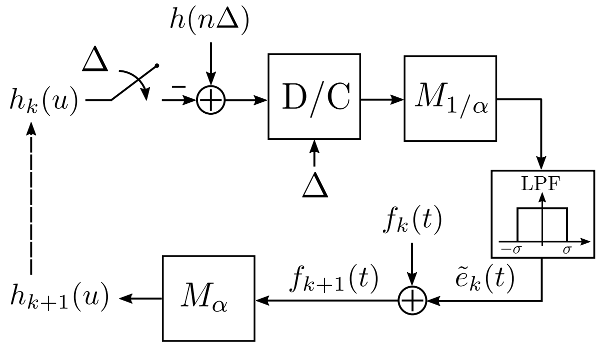


Fig. 3. Block diagram representation of the iterative amplitude sampling reconstruction (IASR) algorithm.

The function $f_\Delta = M_{1/\alpha} h_\Delta$ is nonbandlimited since we have assumed that h_Δ is bandlimited. To obtain a bandlimited approximation to $f(t)$ we then apply a lowpass filter to f_Δ, to obtain the approximation $\tilde{f(t)}$.

Iteratively applying this process results in the iterative algorithm illustrated in Fig. 3, which we refer to as the iterative amplitude sampling reconstruction algorithm (IASR). The focus is placed on reconstructing the amplitude-time function h from the samples $\{h(n\Delta)\}_{n\in\mathbb{Z}}$. The iteration begins with the original uniform samples $h(n\Delta)$ with the objective of iteratively reducing the error $||\tilde{e}_k(t)||_2$.

3.2. Simulations

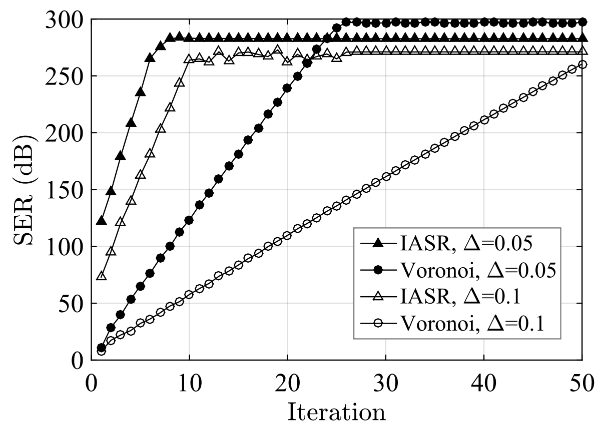
Since amplitude sampling can be regarded as a form of nonuniform time sampling, in our simulations we compare the IASR algorithm to reconstruction based directly on nonuniform time sampling of bandlimited signals. In particular, we compare IASR against the method proposed in [12, Theorem 8.13] referred to as the Voronoi method. It is straightforward to show that the sequence of time instants generated by amplitude sampling satisfies the requirements of the Voronoi method for an appropriate choice of the parameters. In considering the required nonuniform sampling density, we use the definition of sampling density formalized in [13].

We consider as input signals bandlimited noise with frequency components occupying the interval up to σ rad/s. The parameters are chosen such that the the sampling density should be greater or equal than π/σ, which in this case, corresponds to the Landau rate [14]. We choose as a measure of approximation error the signal-to-error ratio (SER) expressed as

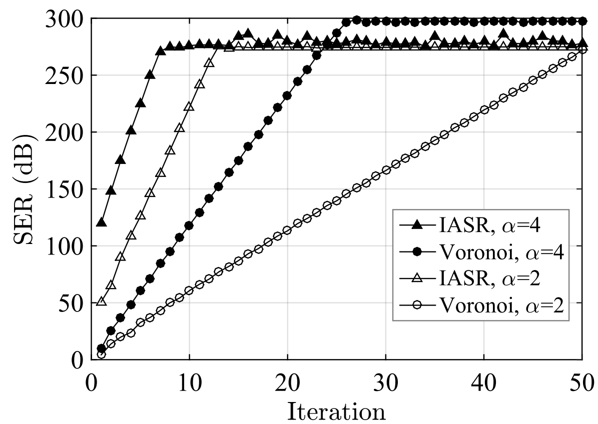
$$\text{SER} = 10\log_{10}\Big(\frac{||f||_2^2}{||f-f_k||_2^2}\Big) \qquad (5)$$

where f_k is the k-th iteration.

IASR requires two functions to be initialized that we set to zero, i.e. $h_0(u) \equiv 0$ and $f_0(t) \equiv 0$ (see Fig. 3). Similarly, we consider the zero function as the initial function in the



(a)



(b)

Fig. 4. Performance comparison between IASR and the Voronoi method for a broadband input signal bandlimited and bounded; (a) Δ is changed while α is fixed; (b) α is changed while Δ is fixed.

Voronoi method case. Note that the first iteration f_1 in the IASR algorithm corresponds exactly to $\tilde{f}$ in Fig. 2. Therefore, as shown in [9], the approximation error of the first iteration of IASR will be significantly lower than the Voronoi method.

Fig. 4 shows the performance when either α or Δ is changed with the other fixed. This corresponds to modifying the number of level crossings per unit of time either by modifying the separation between amplitude levels or by modifying the slope of the ramp. For example by reducing Δ or by increasing α the sampling density increases which results in an improvement of the performance of both methods. Clearly the rate of convergence of the Voronoi method is more sensitive than IASR to the sampling density. For IASR, the rate of convergence doesn't appear to be highly sensitive to the change in parameters. However it is the first iteration that achieves a significantly better approximation for a larger

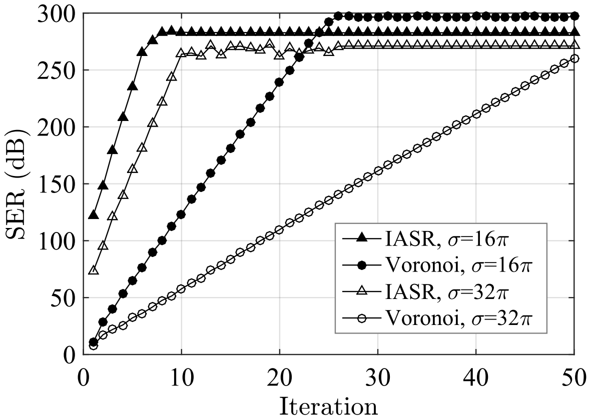


Fig. 5. Performance comparison between IASR and the Voronoi method when the bandwidth σ is changed, and α and Δ are fixed.

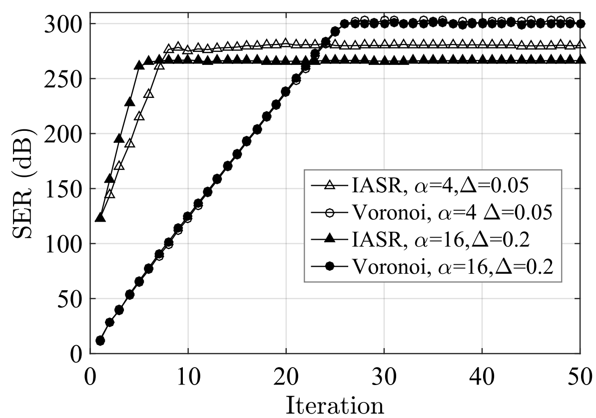


Fig. 6. Performance comparison between IASR and the Voronoi method when the sampling density is fixed and α and Δ are changed.

sampling density. If we now fix α and Δ and reduce the bandwidth of the input signal the oversampling ratio effectively increases and as Fig. 5 illustrates, the performance improves.

In order to isolate the effect of modifying the sampling density when changing parameters, Fig. 6 shows results for a fixed sampling density. In order to do so, the bandwidth is fixed, but both the values of Δ and α are adjusted accordingly in order to maintain the same sampling density. Clearly, the behavior of the Voronoi method remains the same since its convergence rate primarily depends on the sampling density and, in particular, on the maximal value between consecutive sampling instants. It is important to note that the performance of IASR improves as α is increased. The underlying reason is likely due to the increase of the difference $\alpha - A\sigma$, which results in a faster decay in the spectrum of $H(\omega)$.

We have also seen empirically that when the sampling density is significantly large, the Voronoi method presents a faster rate of convergence. However, the more the sampling density approaches the Landau rate, the more evident the faster rate of convergence of IASR is compared to the Voronoi method. In other words, when the sampling instants become more and more sparsely distributed, IASR performs significantly better.

Overall, IASR performs better than the Voronoi method in terms of rate of convergence when the sampling density approaches the Landau rate. The Voronoi method is more sensitive than IASR to changes in the sampling density. The performance of IASR is not only driven by the sampling density but by an increase in the difference $|\alpha| - A\sigma$. In the same manner, it is reported in [15] that a scaling in the input signal to reduce its amplitude range produces an increase in the speed of convergence as well. These results also suggest that IASR more effectively exploits the inherent structure of the function h than the Voronoi method.

4. CONCLUSIONS

We have presented an iterative algorithm to recover a bandlimited signal from samples generated by amplitude sampling. In most situations, it appears to outperform the Voronoi method for reconstruction from nonuniform time samples, and particularly when the sampling instants are sparsely separated.

5. REFERENCES

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