But from the usual partial fraction expansion into distinct linear factors,

$$C_\alpha^{-1} = W'(x_\alpha), \qquad 1 \leq \alpha \leq 2N.$$

Thus (9) assumes the form

$$A_\alpha = \frac{W'(x_\alpha) + W'(x_{\alpha+N})}{W'(x_\alpha)\, W'(x_{\alpha+N})}$$

$$B_\alpha = -\frac{x_\alpha W'(x_\alpha) + x_{\alpha+N} W'(x_{\alpha+N})}{W'(x_\alpha)\, W'(x_{\alpha+N})} \tag{10}$$

and our problem is reduced to the evaluation of the numerators and denominator of A_α and B_α.

Recalling the definition of $P_\alpha(x)$ (see (4)) we see that

$$W'(x_\alpha) = (x_\alpha - x_{\alpha+N}) P_\alpha(x_\alpha)$$

$$W'(x_{\alpha+N}) = -(x_\alpha - x_{\alpha+N}) P_\alpha(x_{\alpha+N}). \tag{11}$$

Thus

$$W'(x_\alpha) + W'(x_{\alpha+N}) = (x_\alpha - x_{\alpha+N}) \sum_{j=1}^{2N-2} a_{j\alpha}(x_\alpha^j - x_{\alpha+N}^j) \tag{12}$$

and

$$x_\alpha W'(x_\alpha) + x_{\alpha+N} W'(x_{\alpha+N}) = (x_\alpha - x_{\alpha+N}) \sum_{j=0}^{2N-2} a_{j\alpha}(x_\alpha^{j+1} - x_{\alpha+N}^{j+1}). \tag{13}$$

The simple factorization

$$x_\alpha^{k+1} - x_{\alpha+N}^{k+1} = (x_\alpha - x_{\alpha+N}) R_{k,\alpha}$$

where

$$R_{k,\alpha} = \sum_{i=0}^{k} x_\alpha^{k-i} x_{\alpha+N}^{i} \tag{14}$$

allows us to write (12) and (13) as

$$W'(x_\alpha) + W'(x_{\alpha+N}) = (x_\alpha - x_{\alpha+N})^2 \sum_{j=1}^{2N-2} a_{j\alpha} R_{j-1,\alpha} \tag{15}$$

and

$$x_\alpha W'(x_\alpha) + x_{\alpha+N} W'(x_{\alpha+N}) = (x_\alpha - x_{\alpha+N})^2 \sum_{j=0}^{2N-2} a_{j\alpha} R_{j,\alpha} \tag{16}$$

respectively.

Now let us look at the denominator of A_α and B_α (10). From (11) and (4)

$$W'(x_\alpha)\, W'(x_{\alpha+N}) = -(x_\alpha - x_{\alpha+N})^2 P_\alpha(x_\alpha) P_\alpha(x_{\alpha+N})$$

$$= -(x_\alpha - x_{\alpha+N})^2 \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} Q_\beta(x_\alpha)\, Q_\beta(x_{\alpha+N}).$$

The identity

$$Q_\beta(x_\alpha)\, Q_\beta(x_{\alpha+N}) = \pi_\alpha Q_\alpha(\sigma_\beta) + \pi_\beta Q_\beta(\sigma_\alpha) - 2\pi_\alpha \pi_\beta$$

then immediately yields

$$W'(x_\alpha)\, W'(x_{\alpha+N}) = -(x_\alpha - x_{\alpha+N})^2\, \Psi_\alpha \tag{17}$$

where Ψ_α is given by (7).

Equations (15)-(17) then reduce to (6) (see (10)). It remains but to evaluate $R_{k,\alpha}$ in terms of σ_i and π_j. But this formula is an immediate consequence of the following lemma.

Lemma

If $\sigma = \alpha + \beta$ and $\pi = \alpha\beta$, then

$$\sum_{i=0}^{k} \alpha^{k-i}\beta^{i} = 2^{-k} \sum_{i=0}^{[1/2k]} \binom{k+1}{2i+1} \sigma^{k-2i}(\sigma^2 - 4\pi)^i. \tag{18}$$

Proof: Let $r = \sqrt{\sigma^2 - 4\pi}$. Then $\alpha = \frac{1}{2}(\sigma - r)$ and $\beta = \frac{1}{2}(\sigma + r)$. Substituting these values of α and β on the right-hand side of the identity

$$\sum_{i=0}^{k} \alpha^{k-i}\beta^{i} = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}$$

and using the binomial theorem yields (18).

An Algorithm for the Numerical Evaluation of the Hankel Transform

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***Abstract*—A procedure is proposed for the numerical evaluation of the Hankel (Fourier-Bessel) transform of any integer order using the FFT algorithm. The basis for the procedure is the "projection-slice" theorem associated with the two-dimensional Fourier transform.**

In a variety of applications, the need arises for the numerical evaluation of the Hankel transform (alternatively referred to as the Fourier-Bessel transform). For example, in ocean acoustics, the reflected pressure field from a horizontally stratified bottom and the plane wave reflection coefficient are related through the Hankel transform [1]. Other common areas in which similar relationships arise are in optics [2] and in molecular biology [3].

The Hankel transform can be (and often is) interpreted in terms of the two-dimensional Fourier transform. Specifically, let $f(x, y)$ and $F(\mu, \nu)$ denote a two-dimensional function and its Fourier transform so that

$$F(\mu, \nu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, e^{j\mu x} e^{j\nu y} dx\, dy \tag{1}$$

or, with $f(x, y)$ and $F(\mu, \nu)$ expressed in polar coordinates,

$$\mathfrak{F}(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \mathfrak{F}(r, \theta) \exp\{j[\cos(\theta - \phi)]\, r\rho\}\, r\, dr\, d\theta \tag{2}$$

where θ is measured relative to the x-axis and ϕ is measured relative to the μ-axis. If $\mathfrak{F}(r, \theta)$ is of the form

$$\mathfrak{F}(r, \theta) = g(r) e^{jm\theta} \tag{3}$$

where m is an integer, then (2) reduces to [2]

$$\mathfrak{F}(\rho, \phi) = (j)^m G(\rho) e^{jm\phi} \tag{4}$$

where

$$G(\rho) = \int_0^{\infty} J_m(r\rho)\, g(r)\, r\, dr. \tag{5}$$

The integral relationship of (5) corresponds to the Hankel transform of order m [4]. From (4), we see that it is equal to $(j)^{-m} e^{-jm\phi_0}$ times a

Manuscript received October 3, 1977. This work was supported in part by Advanced Research Projects Agency, monitored by ONR under Contract N00014-75-C-0951-NR, and in part by ONR Contract N00014-77-C-0196.

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0018-9219/78/0200-0264$00.75

slice at angle ϕ_0 through the two-dimensional transform $\mathfrak{F}(\rho, \phi)$. Our proposed method of numerically evaluating (5) is based on the "projection-slice" theorem for the two-dimensional Fourier transform. This theorem states that the one-dimensional transform of a *projection* of $f(x, y)$ at any angle is a *slice* at the same angle of $F(\mu, \nu)$ [5]. For example, referring to (1), let us consider the slice in $F(\mu, \nu)$ corresponding to $\nu = 0$, or equivalently, $\mathfrak{F}(\rho, \phi)$ for $\phi = 0$. Then

$$F(\mu, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\mu x} p(x)\, dx \tag{6}$$

where

$$p(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy \tag{7}$$

is the projection of $f(x, y)$ onto the x-axis. Thus from (6) and (4), we can write that

$$G(\rho) = \frac{j^{-m}}{2\pi} \int_{-\infty}^{+\infty} e^{j\rho x} p(x)\, dx. \tag{8}$$

Thus comparing (5) and (8), it follows that the mth-order Hankel transform can be equivalently expressed (and calculated) as j^{-m} times the one-dimensional Fourier transform of the projection $p(x)$.

The two basic computational steps in evaluating (5) using this approach are the evaluation of the projection $p(x)$ (7) and the evaluation of the one-dimensional Fourier transform (8). Let us assume that $G(\rho) = 0$, $|\rho| \geqslant R_0$. Then, from (7), $p(x)$ is bandlimited, and consequently, by virtue of the sampling theorem,

$$j^m G(\rho) = \frac{\Delta x}{2\pi} \sum_{k=-\infty}^{+\infty} p(k\Delta x) e^{j\rho k \Delta x} \tag{9}$$

provided that $\Delta x < \pi/R_0$. If we consider calculating $G(\rho)$ at N equally spaced values $\Delta\rho = (1/N)(2\pi/\Delta x)$, then

$$j^m G(k\Delta\rho) = \frac{\Delta x}{2\pi} \sum_{n=0}^{N-1} \left[\sum_{r=-\infty}^{+\infty} p[(n + rN)\Delta x] \right] \exp\left(j \frac{2\pi}{N} nk\right). \tag{10}$$

Thus, $G(k\Delta\rho)$, $k = 0, 1, \cdots, N-1$, is proportional to the discrete Fourier transform of the samples of $p(x)$, aliased in x. If the samples of $p(x)$ represent a finite-length sequence of length $\leqslant (N\Delta x)$, then (10) reduces to

$$j^m G(k\Delta\rho) = \frac{\Delta x}{2\pi} \sum_{n=0}^{N-1} p(n\Delta x) \exp\left(j \frac{2\pi}{N} nk\right). \tag{11}$$

Both (10) and (11) correspond to the discrete Fourier transform, and consequently they can be evaluated directly using the one-dimensional FFT.

The calculation of samples of $p(x)$ is somewhat less direct. Equation (7) can equivalently be written as

$$p(x) = 2 \int_0^{+\infty} g\left(\sqrt{x^2 + y^2}\right) V_m \left(\frac{x}{\sqrt{x^2 + y^2}}\right) dy \tag{12a}$$

$$p(x) = 2 \int_x^{+\infty} g(r) \frac{r}{\sqrt{r^2 - x^2}} V_m \left(\frac{x}{r}\right) dr \tag{12b}$$

$$p(x) = 2x \int_0^{\pi/2} g\left(\frac{x}{\cos\theta}\right) \frac{\cos m\theta}{\cos^2\theta}\, d\theta. \tag{12c}$$

where $V_m(\cdot)$ is the mth-order Chebyshev polynomial. Equations (12) incorporate the fact that since $f(r, \theta)$ is conjugate antisymmetric in θ, only its real part contributes to $p(x)$. We have found it most convenient to calculate $p(x)$ through the use of (12a). Specifically, we note that since $f(x, y)$ is bandlimited,

$$\int_{-\infty}^{+\infty} f(x, y)\, dy = \Delta y \sum_{k=-\infty}^{+\infty} f(x, k\Delta y) \tag{13}$$

provided only that $\Delta y < 2\pi/R_0$. Equation (13) is basically a consequence of the fact that for a bandlimited function sampled at one-half the Nyquist rate or higher, its integral is directly proportional to the sum of its samples. Thus, $p(n\Delta x)$ as required in (10) or (11) is

$$p(n\Delta x) = \Delta y \sum_{k=-\infty}^{+\infty} g\left(\sqrt{(n\Delta x)^2 + (k\Delta y)^2}\right) V_m \left(\frac{n\Delta x}{\sqrt{n^2\Delta x^2 + k^2 \Delta y^2}}\right). \tag{14}$$

Equations (10) and (14) together provide an exact expression for the numerical calculation of $G(k\Delta\rho)$ provided only that $G(\rho) = 0$, $|\rho| > R_0$. If this is not the case, then (10) will compute samples of $G(\rho)$ aliased in ρ, i.e.,

$$\sum_{q=-\infty}^{+\infty} G[\Delta\rho(k + qN)] \tag{15}$$

and an integration rule more complex than (13) must be used to calculate $p(x)$.

To evaluate (14), we assume that $g(\sqrt{x^2 + y^2})$ is known on a rectangular grid in the x-y plane. In many practical cases of interest, including the one that motivated our consideration of this method, $g(r)$ is generally available as samples in r. In this case, evaluation of (14) requires an interpolation of samples of $g(r)$ to the sample points on the rectangular grid. Under the assumption that $G(\rho) = 0$, $|\rho| > R_0$, this is the only step in the procedure in which an approximation is required.

The above procedure has been successfully applied to a number of trial examples. Because of its apparent accuracy and efficiency, it is presently being utilized and explored further in the context of seabed acoustics.

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