DIGITAL PRE-COMPENSATION FOR FAULTY D/A CONVERTERS: THE "MISSING PIXEL" PROBLEM

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ABSTRACT

In some contexts, DACs fail in such a way that specific samples are dropped. For example in flat-panel video displays, one of the pixel LEDs can malfunction and get permanently set to a particular value. We refer to this as the "missing pixel" problem. Under certain conditions, it may be possible to compensate for the dropped sample by pre-processing the digital signal. This paper describes a number of such compensation strategies. Each strategy is analyzed and results from numerical simulations are presented. Of particular interest is the relationship between compensation and the class of discrete prolate spheroidal sequences.

1. INTRODUCTION

The "missing pixel" problem is of practical interest and some ad hoc solutions have been proposed, [1, 2]. These patents use a scheme where neighboring pixels are brightened in order to compensate. The idea is based on the fact that the missing pixel looks dark, so making the surrounding pixels brighter reduces the visual distortion. Though several weightings are proposed, no theory is developed. In this paper, we present a more rigorous treatment of the problem and propose solutions which are optimal. The following analysis and solutions have also been presented in [3, 4].

In one-dimension, the faulty DAC can be mathematically represented as in Figure 1(a). We adhere to the standard paradigm for digital-to-analog conversion through an ideal low-pass filter. In addition, we assume the original continuous-time signal, $x(t)$, is at least slightly oversampled. We assume specifically that $1/T_s = R\Omega_c/\pi$, where $x(t)$ is band-limited to Ω_c and $R > 1$ is the oversampling ratio. We denote the ratio π/R by γ.

The dropped sample is represented as multiplication by $(1 - \delta[n])$ that sets $x[0] = 0$. Because of the dropped sample, the reconstructed signal, $\hat{r}(t)$, is a distorted version of the desired reconstruction, $r(t)$. Compensation is portrayed as a signal $c[n]$ that is added to $x[n]$. In general, compensation could be some complicated function of the dropped sample and neighbors. In our development, we restrict compensation to be an affine transformation.

We use the squared-$\mathcal{L}_2$ energy of the error signal, $e(t) = \hat{r}(t) - r(t)$, as the error metric:

$$\mathcal{E}^2 = \int_{-\infty}^{\infty} |e(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{r}(t) - r(t)|^2 dt \tag{1}$$

The problem can equivalently be cast directly in the discrete-time domain. Specifically, it is straightforward to show that Figure 1(a) can be equivalently represented by Figure 1(b). $H(e^{j\omega})$ is an ideal low-pass filter with cutoff $\gamma = \Omega_c/T_s = \pi/R$. Additionally, we incorporate the missing sample into the compensation as a constraint on $c[n]$, specifically $c[0] = -x[0]$. We also equivalently use the squared-ℓ_2 energy of $e[n]$ as an error metric:

$$\varepsilon^2 = \sum_{n=-\infty}^{\infty} |e[n]|^2 = \sum_{-\infty}^{\infty} |\hat{r}[n] - x[n]|^2 \tag{2}$$

In Figure 1(b), the error signal, $e[n]$, can be expressed

$$e[n] = x[n] - h[n] * (x[n] + c[n]) \tag{3}$$

Since $X(e^{j\omega})$ is band-limited to γ, our error (2) reduces to

$$\varepsilon^2 = \int_{<2\pi>} |H(e^{j\omega})C(e^{j\omega})|^2 d\omega \tag{4}$$

Thus, minimizing the squared-ℓ_2 error is equivalent to minimizing the energy of $C(e^{j\omega})$ in the band $[-\gamma,\gamma]$, i.e. in the passband of the filter. This also implies that $x(t)$ must be at least slightly oversampled for compensation, otherwise $H(e^{j\omega})$ could not be designed with a high-frequency stop-band.

Affine pre-compensation can in general alter any arbitrary set of samples in $x[n]$. For clarity in the exposition, we focus on symmetric compensation, where $(N - 1)/2$ neighboring samples on either side of the dropped sample are altered. Where the extension to the more general case is obvious, we note the structure of the asymmetric solution.

In Section 2 of this paper, we discuss the ideal solution in which infinitely many samples may be adjusted. In Section 3, we consider the case in which only a finite number of samples may be adjusted, and we derive the optimal solution. Section 4 and 5 present two alternatives, an approximation to the optimal solution and an iterative algorithm which converges to the optimal solution.

2. PERFECT COMPENSATION

If $c[n]$ had no frequency component outside $|\omega| > \pi - \gamma$ while meeting the constraint $c[0] = -x[0]$, it would perfectly compensate with zero error. There are an infinite number of signals that

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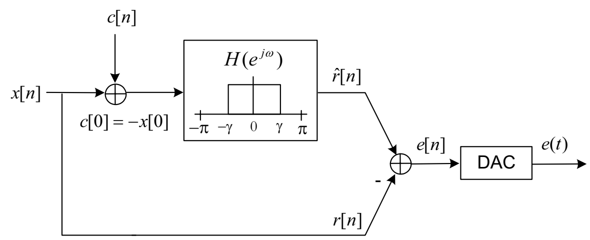
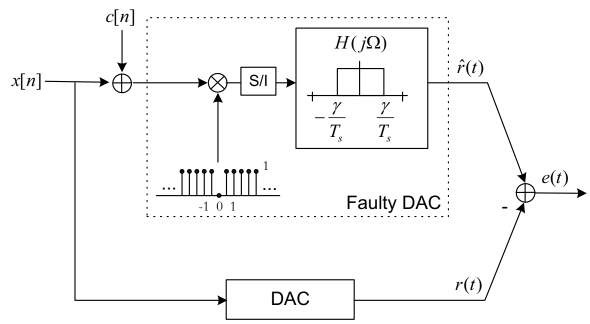


(a) Problem Representation



(b) Reduced Representation

Fig. 1. Representation of faulty DAC with compensation

meet this criteria. For example, we can simply choose

$$c_{\inf}[n] = -x[0](-1)^n \tag{5}$$

This solution only requires in theory that $R = 1 + \epsilon$, where ϵ is non-zero but otherwise arbitrarily small. It can be shown that all other perfect compensation solutions are signals band-limited to $|\omega| < \gamma$ multiplied by $c_{\inf}[n]$. Of these choices, the minimum energy solution is

$$c_{\text{sinc}}[n] = -x[0](-1)^n \frac{\sin(\pi-\gamma)n}{(\pi-\gamma)n} \tag{6}$$

Unfortunately, all of these signals, although resulting in zero error, have infinite length, making them impractical to implement.

3. CONSTRAINED MINIMIZATION

Because of the impracticality of perfect compensation, we focus next on finite-length compensation. We assume that the compensation signal, $c[n]$, is non-zero only for $n \in \mathcal{N}$, where $\mathcal{N}$ is the finite set of points to be adjusted. For simplicity in the presentation, we focus on symmetric compensation, $\mathcal{N} = [-\frac{N-1}{2}, \frac{N-1}{2}]$, although the derivation is general for any set $\mathcal{N}$.

We minimize $\varepsilon^2 = \sum_{n=-\infty}^{\infty}(\sum_{m\in\mathcal{N}} c[m]h[n-m])^2$ subject to the constraint $g = c[0]+x[0] = 0$ using the method of Lagrange multipliers. The minimization produces $N+1$ linear equations that have a unique solution, $c_{\text{opt}}[n]$, corresponding to the optimal compensation signal for the given value of N. We can write the equations in block matrix form as

$$\begin{bmatrix} \mathbf{\Theta}_\gamma & -\frac{1}{2}\delta \\ \delta^{\mathbf{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c_n} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -x_0 \end{bmatrix} \tag{7}$$

λ is the Lagrange multiplier and δ is a vector with all zero entries except for a 1 as the center element. $\mathbf{\Theta}_\gamma$ is the autocorrelation matrix for $h[n]$. Because $h[n]$ is an ideal low-pass filter, $\mathbf{\Theta}_\gamma$ is a symmetric, Toeplitz matrix with entries $\mathbf{\Theta}_\gamma(i,j) = h[i-j] = \frac{\sin\gamma[i-j]}{\pi[i-j]}$. We prove in Section 4 that $\mathbf{\Theta}_\gamma$ is invertible. For now, assuming we can invert $\mathbf{\Theta}_\gamma$, the optimal solution is

$$c_{\text{opt}}[n] = -\frac{x[0]}{\theta_c^{-1}} \mathbf{\Theta}_\gamma^{\mathbf{-1}} \delta \tag{8}$$

$\theta_c^{-1} = \mathbf{\Theta}_\gamma^{\mathbf{-1}}\left(\frac{N-1}{2}, \frac{N-1}{2}\right)$, the center element of the inverse matrix. We refer to the algorithm represented by (8) as Constrained Minimization (CM). Exploiting the Toeplitz, symmetric structure, we can use a Levinson recursion to invert $\mathbf{\Theta}_\gamma$. CM can thus be implemented with $O(N^2)$ multiplications and $O(N)$ memory.

Figure 2(a) illustrates ε^2 as a function of N. The graph shows that ε^2 decreases approximately exponentially in N. Since the CM algorithm generates the optimal solution, the error curves shown in Figure 2(a) serve as a baseline for performance of other finite-length choices for $c[n]$.

There is a limited set of parameters γ and N for which the problem is well-conditioned. Beyond $\varepsilon^2 = 10^{-9}$, the solution becomes numerically unstable beyond the precision of MATLAB. Conditioning problems would be even more pronounced in fixed-point DSP systems, but the inversion can be done off line on a computer with arbitrarily high precision, since once $c_{\text{opt}}[n]$ is found it can be stored and retrieved when the algorithm needs to be implemented. Also, in most contexts, an error of $10^{-9} = -180$dB, compared to the signal, is more than sufficient.

4. DISCRETE PROLATE APPROXIMATION

With CM, we construct a finite-length compensation signal directly from the imposed constraints. Alternatively, we can start with the infinite-length signal, $c_{\inf}[n] = -x[0](-1)^n$, and truncate it through appropriate windowing. From this perspective, the problem then becomes one of designing a finite-length window, $w[n]$, such that

$$c[n] = w[n]c_{\inf}[n] \tag{9}$$

has minimum energy in the frequency band $|\omega| < \gamma$. Since $C(e^{j\omega}) = W(e^{j\omega}) * C_{\inf}(e^{j\omega})$, we design the window $w[n] \in \ell_2(-\frac{N-1}{2}, \frac{N-1}{2})$, to maximize the concentration ratio

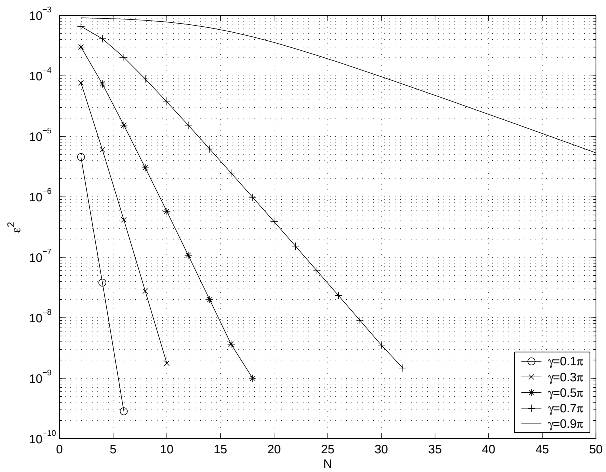
$$\alpha(N,W) = \frac{\int_{-W=-\pi+\gamma}^{W=\pi-\gamma} |W(e^{j\omega})|^2 d\omega}{\int_{-\pi}^{\pi} |W(e^{j\omega})|^2 d\omega} \tag{10}$$

Slepian, Landau, and Pollak solved this problem in [5, 6, 7] through the development of discrete prolate spheroidal sequences (DPSS). Using variational methods [7] shows that the signal $w[n]$ that maximizes the concentration is an eigenvector of the $N \times N$ symmetric, positive-definite, Toeplitz matrix, $\mathbf{\Theta_W}$, with elements

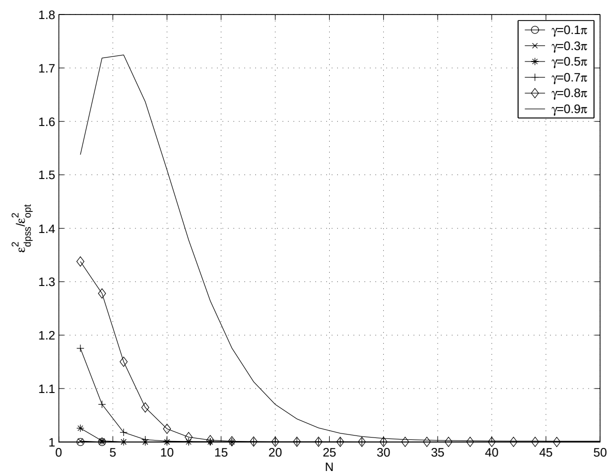
$$\mathbf{\Theta_W}[n,m] = \frac{\sin 2W(m-n)}{\pi(m-n)} \tag{11}$$

$$m,n = -(N-1)/2, \ldots -1, 0, 1, \ldots, (N-1)/2$$

If $W = \gamma$, we obtain $\mathbf{\Theta}_\gamma$, the same matrix as in Section 3. By the spectral theorem, the eigenvectors, $v_i^W[n]$, are real and orthogonal with associated real, positive eigenvalues, λ_i^W. In addition, [7] proves that these particular eigenvalues are always distinct. The eigenvectors, $v_i^W[n]$, are time-limited versions of discrete prolate



(a) Squared-error of $c_{\rm opt}[n]$



(b) Error gain of $c_{\rm dpax}[n]$ over $c_{\rm opt}[n]$

Fig. 2. Error performance of CM and DPAX

spheroidal sequences (DPSS). They form a finite orthonormal basis for $\ell_2(-\frac{N-1}{2}, \frac{N-1}{2})$, [7].

The first DPSS, $v_1[n]$, solves the concentration problem. The maximum concentration is the eigenvalue, λ_1. Consequently, the optimal window in our formulation is $v_1^{\pi-\gamma}[n]$. Modulating $v_1^{\pi-\gamma}[n]$ up to π and scaling it to meet the constraint, $c[0] = -x[0]$, provides a potential compensation signal

$$c_{\rm dpax}[n] = -\frac{x[0]}{v_1^{\pi-\gamma}[0]}(-1)^n v_1^{\pi-\gamma}[n] \qquad (12)$$

Every DPSS has a dual symmetric partner. In particular,

$$v_{N+1-i}^{W}[n] = (-1)^n v_i^{\pi-W}[n] \qquad (13)$$

The eigenvalues are related, $\lambda_{N+1-i}^{W} = \lambda_i^{\pi-W}$. [3] provides a comprehensive proof of this property using filter-banks. Duality implies that the compensation signal, $c_{\rm dpax}[n]$ in (12), can be equivalently expressed as

$$c_{\rm dpax}[n] = -\frac{x[0]}{v_N^{\gamma}[0]} v_N^{\gamma}[n] \qquad (14)$$

Independent of which DPSS is used to express it, we denote this solution the Discrete Prolate Approximation (DPAX). For asymmetric compensation, the solution is the same, except the scaling is relative to the dropped sample, $v_N^{\gamma}[k]$, for $k \neq 0$.

It should be clear that $c_{\rm dpax}[n]$ is not equivalent to the CM solution, $c_{\rm opt}[n]$. The window formulation starts with a finite-energy signal, optimizes for that energy, and then scales to meet the $c[0] = -x[0]$ constraint. CM does not begin an energy constraint, thus it finds the optimal solution.

The exact relationship between $c_{\rm dpax}[n]$ and $c_{\rm opt}[n]$ can be found by decomposing $c_{\rm opt}[n]$ in the DPSS basis, $\{v_i^{\gamma}[n]\}$. The time-limited DPSS form the orthonormal eigenvector basis that diagonalizes $\mathbf{\Theta}_\gamma = \mathbf{V\Lambda V^T}$. Since each DPSS eigenvalue is distinct, [7], this proves that $\mathbf{\Theta}_\gamma$ is invertible and that $c_{\rm opt}[n]$ exists. $c_{\rm opt}[n]$ is proportional to $\mathbf{\Theta}_\gamma^{-1}$, so it can be expressed as

$$c_{\rm opt}[n] = -\frac{x[0]}{\theta_c^{-1}}\left(\lambda_1^{-1}\beta_1 v_1^{\gamma}[n] + \cdots + \lambda_N^{-1}\beta_N v_N^{\gamma}[n]\right) \qquad (15)$$

$\beta_i = v_i^{\gamma}[0]$ and θ_c^{-1} is the middle element of $\mathbf{\Theta}_\gamma^{-1}$, which can be expressed in the DPSS basis as $\theta_c^{-1} = \sum_{i=1}^{N}\lambda_i^{-1}(v_i^{\gamma}[0])^2$. The eigenvalues, λ_i, are distributed between 0 and 1. The expression for the optimal solution depends on the reciprocals $1/\lambda_i$, so the eigenvector with the smallest eigenvalue, $v_N^{\gamma}[n]$, will dominate. Since scaling this vector produces $c_{\rm dpax}[n]$, DPAX can be interpreted as a first-order approximation to $c_{\rm opt}[n]$.

Figure 2(b) plots the gain in error, $\varepsilon^2_{dpss}/\varepsilon^2_{opt}$, due to DPAX. The gain becomes negligible as N increases and γ decreases. Additionally, DPAX does not suffer from the same ill-conditioning problems as CM, so, by increasing N, it can achieve values of ε^2 in the range of $\varepsilon^2 = 10^{-20}$, i.e. about ten orders of magnitude smaller than that using CM.

The near-optimal performance of the DPAX solution is explained by the eigenvalue distribution of $\mathbf{\Theta}_\gamma$. As N increases and γ decreases, the reciprocal of the smallest eigenvalue, $1/\lambda_N$, increasingly dominates the reciprocals of the other eigenvalues. In (15), $v_N^{\gamma}[n]$ dominates the other terms, making $c_{\rm dpax}[n]$ a tighter approximation

The DPAX solution can be computed directly as the first eigenvector of $\mathbf{\Theta}_{\pi-\gamma}$. However, the eigenvalues of $\mathbf{\Theta}_{\pi-\gamma}$ are so clustered around 0 or 1 that they are effectively degenerate when finite machine arithmetic is used. Fortunately, the time-limited DPSS are also eigenvectors of the symmetric, tri-diagonal matrix $\rho_{\pi-\gamma}$

$$\rho_{\pi-\gamma}[i,j] = \begin{cases} \frac{1}{2}i(N-i) & j = i-1 \\ \left(\frac{N-1}{2}-i\right)^2\cos 2(\pi-\gamma) & j = i \\ \frac{1}{2}(i+1)(N-1-i) & j = i+1 \\ 0 & \text{otherwise} \end{cases} \qquad (16)$$
$$i,j = -(N-1)/2, \ldots -1, 0, 1, \ldots (N-1)/2$$

which has eigenvalues that are well spread, [8]. Accordingly, the first DPSS can then be computed using standard routines like the iterative power method. DPAX is a powerful alternative algorithm to CM. Its performance is nearly optimal, it is less complex, and it has fewer stability problems.

5. ITERATIVE MINIMIZATION

In this section, as an alternative to the two closed-form algorithms, we develop an iterative solution in the class of projection-onto-convex sets (POCS). Figure 3(a) is a block diagram of the algorithm, denoted Iterative Minimization (IM). Each iteration consists of three sequential projections, P_B onto $\ell_2(\pi-\gamma)$, P_D onto

$\ell_2(-N/2, N/2)$, and P_0 onto the hyperplane defined by the constraint $c[0] = -x[0]$.

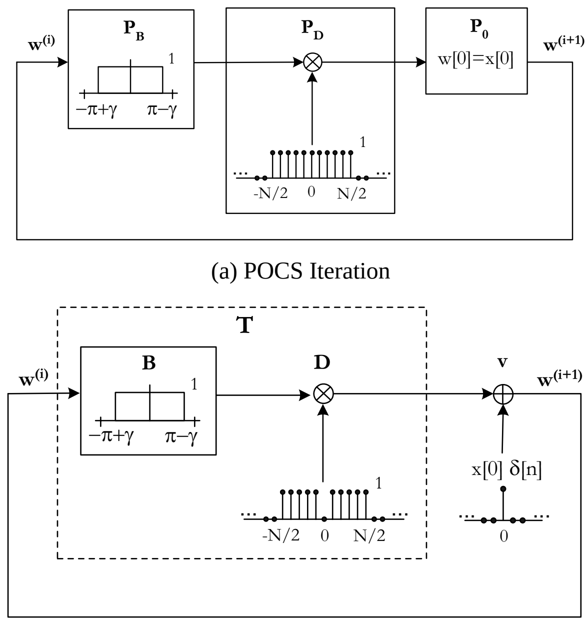
The iteration can be proved to converge uniquely to $(-1)^n c_{\text{opt}}[n]$. To facilitate proofs, we represent the projections in Figure 3(a) in terms of the affine transformation of Figure 3(b). We first prove that the linear operator, T, is strictly non-expansive, i.e. for $\mathbf{w_1} \neq \mathbf{w_2}$, $T(\mathbf{w_1} - \mathbf{w_2}) < \mathbf{w_1} - \mathbf{w_2}$, using an argument based on the fact that B, band-limiting, strictly reduces the energy in the time-limited signal, $\mathbf{w^{(i)}}$. A detailed proof is not presented here; [3] develops the argument fully. A strictly non-expansive T implies that IM converges strongly to a unique fixed-point, [3].

Next, we show that the fixed-point is $\mathbf{w}^* = (-1)^n c_{\text{opt}}[n]$ by direct substitution. In Figure 3(a), P_B and P_D can be conglomerated into a Toeplitz matrix, $\mathbf{\Theta}_{\pi-\gamma}$. Using the decomposition of $c_{\text{opt}}[n]$, as per (15), and the duality of the DPSS, $P_B P_D \mathbf{w}^*$ can be manipulated into two terms. β_i is defined as in (15). The first term is the fixed-point. The other, which is composed of residual terms, is a decomposition of an impulse, $\delta[n]$, into the DPSS basis. Projection with P_0 removes this term and returns $\mathbf{w}^* = (-1)^n c_{\text{opt}}[n]$.

$$\frac{-x[0]}{\theta_c^{-1}}(-1)^n \left(\beta_1 \lambda_1^{-1}(v_1^\gamma[n]) + \cdots + \beta_N \lambda_N^{-1}(v_N^\gamma[n])\right) \quad (17)$$
$$+\frac{-x[0]}{\theta_c^{-1}}(-1)^n \left(\beta_1(v_1^\gamma[n]) + \cdots + \beta_N(v_N^\gamma[n])\right)$$

$$= (-1)^n c_{\text{opt}}[n] + \delta[n] \quad (18)$$

Although IM converges to the optimal solution, it has a slow convergence rate. Numerical simulation shows that, as N increases and γ decreases, the convergence rate slows to the point of making IM impractical compared to CM, i.e. convergence requires greater than $O(N^2)$ iterations. Slow convergence is caused by the eigenvalues of $\mathbf{\Theta}_{\pi-\gamma}$ being clustered near 1, so there is minimal change between iterations. Though not studied in this treatment, POCS relaxation techniques could potentially be used to speed up the convergence rate.



(a) POCS Iteration

(b) Affine Iteration

Fig. 3. Iterative Optimal Window Design

6. CONCLUSION

In this paper, we present a number of variations on digital compensation techniques to reduce the effect of dropped samples in DACs. The ideal solution is a perfect, infinite-length compensation signal, $c_{\text{inf}}[n]$. For practical implementation, though, we develop two algorithms to compute the optimal, finite-length solution: CM, a closed-form calculation, and IM, an iterative POCS algorithm. Despite calculating the optimal solution, both algorithms are found to be computationally expensive and ill-conditioned over a large range of parameters.

As an alternative, we develop a compensation strategy using discrete prolate spheroidal sequences. Our solution, termed DPAX, is shown to be a tight, first-order approximation of the optimal, finite-length solution in the DPSS basis. In addition, DPAX is less complex and better conditioned than either CM or IM.

The pre-compensation problem was originally conceived in the context of missing pixels on flat-panel displays. To this end, we implemented a two-dimensional version of the CM solution and applied it to actual images. The resulting compensated images are presented in [4].

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